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# DEVELOPMENT OF COMPUTATIONAL ALGORITMS TO SIMULATE FRICTIONAL CONTACT OSCILLATING PORCESSES

Для разработки вычислительного алгоритма решения динамических задач с трением используется вариационный подход. Дискретизация вариационных задач по времени производилась на основе двух- и трехслойных разностных схем. Построены эквивалентные вариационным неравенствам с недифференцируемыми слагаемыми задачи минимизации, решение которых получено в явном виде.

Для розробки обчислювального алгоритму розв'язання динамічних задач з тертям використовується варіаційний підхід. Дискретизація варіаційних задач за часом проводилася на основі двох- і тришарових різницевих схем. Побудовано еквівалентні варіаційним нерівностям з недиференціруємими складовими завданнями мінімізації, рішення яких отримано в явному вигляді.

Variational approach is applied to develop computational algorithm for solving dynamic problems with friction. Temporal discretization of variational problems was performed using duplex and triplex difference schemes. Minimization problems equivalent to variational inequalities with nondifferentiable components are developed; their solution is obtained in explicit form.

**Introduction.** In related transport industries world practice of perspective brake-building applies current mechanical problem solving for adjusting materials using composite, metal-ceramic and polymeric-asbestos brake blocks[1, 2, 3, 4]. However, in this context, developments of adaptive devices structures, tribology and tribomechanics to modify elastic and dissipative characteristics of brake systems loading are not applied [5, 6]. Moreover, oscillation loading of brake to control friction procedure and to obtain required brake parameters in dynamics is not covered.

Theory hypothesizes that brake gear is a mechanism with rigid segments. Mine rail haulage also ignores control of brake frictional behaviour efficiency. That is idealized dynamic brake model is used; however, inertia as well as elastic and damping properties of segments and contact are neglected leading to a contradiction between resulting in haulage equipment implementation and mine train brake.

**The work objective** is to simulate frictional micro-oscillations using computational experiment to control friction while operating in terms of value and slipping velocity function at the expense of regulating elastic and dissipative characteristics of brake system loading.

**Original material presentation.** Basic oscillating system with two degrees of freedom is specified as analytical model of brake (Fig. 1). The system consists of a shoe with $m$ mass sliding over a wheel with $R$ radius rotating with uniform angular velocity $\omega$, and elastic and damping Voigt element which hardness and viscosity we call $c$ and $b$ respectively. Surface curvature of shoe and wheel is neglected. Constant external force $Q$ is exerted on a shoe pressing it to a wheel. Nominal area of shoe and wheel contact is of square shape with $2a$ and $e$ legs. Actual interaction contact region is discrete consisting of contact patches group. Roughness of contact surfaces results in discreteness.

Dynamic behaviour of the system meets following inequality

$$(m\ddot{x} + (b\dot{x} + cx)\beta_{cc} - (b\dot{y} + cy)\beta_{cs})(\dot{u} - \dot{x}) + \varphi F_y(x,y)(|\dot{u} - U| - |\dot{x} - U|) +$$
$$+ (m\ddot{y} + (b\dot{y} + cy)\beta_{ss} - (b\dot{x} + cx)\beta_{cs} - F_y(x,y) + Q)(\dot{v} - \dot{y}) \geq 0, \quad \forall\{u,v\}. \quad (1)$$

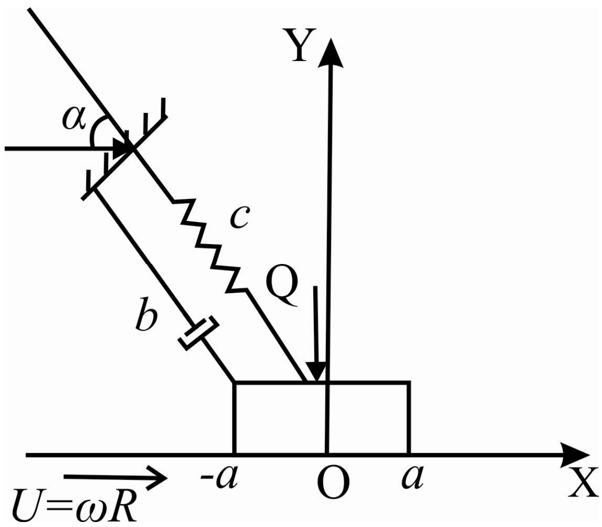


Fig. 1. Design Diagram of Braking Gear Dynamic Model

Perform discretization of variational problem (1). Divide time axis into equivalent segments $[t^{n-1}, t^n]$, $(n = 1,2,...)$. Length of the segments denote by $h$. Then $\{x^n, y^n\}$ will be meant as approximate value $\{x(t), y(t)\}$ in time point $t^n$. Integration formulas for each method discussed below result from inserting expressions for velocities and accelerations in time points under consideration into quasivariational inequalities (1); it is done through motion corresponding to the time period, and motion, velocity, and acceleration determined during previous integration steps:

$$\dot{x}^{n+1} = \alpha^{(1)} x^{n+1} + \sum_{i=0}^{I} (\alpha_i^{(1)} x + \alpha_i^{(2)} \dot{x}^{n-i} + \alpha_i^{(3)} \ddot{x}^{n-i}); \quad (2)$$

$$\ddot{x}^{n+1} = \beta^{(1)} x^{n+1} - \sum_{i=0}^{I} (\beta_i^{(1)} x - \beta_i^{(2)} \dot{x}^{n-i} - \beta_i^{(3)} \ddot{x}^{n-i}), \quad (3)$$

$$\dot{y}^{n+1} = \alpha^{(1)} y^{n+1} + \sum_{i=0}^{I} (\alpha_i^{(1)} y + \alpha_i^{(2)} \dot{y}^{n-i} + \alpha_i^{(3)} \ddot{y}^{n-i}); \quad (4)$$

$$\ddot{y}^{n+1} = \beta^{(1)} y^{n+1} - \sum_{i=0}^{I} (\beta_i^{(1)} y - \beta_i^{(2)} \dot{y}^{n-i} - \beta_i^{(3)} \ddot{y}^{n-i}), \quad (5)$$

where $\alpha_i^{(j)}$ and $\beta_i^{(j)}$ are coefficients determined by $h$ value.

*Triplex Difference Schemes*

Using quasivariational inequality (1) of triplex scheme with balance [7] for temporal integration we obtain:

$$\Bigg( m\frac{x^{n+1}-2x^{n}+x^{n-1}}{h^2}+b\beta_{cc}\frac{x^{n+1}-x^{n-1}}{2h}+c\beta_{cc}(\theta_1 x^{n+1}+\theta_2 x^{n}+\theta_3 x^{n-1})-$$

$$+\Bigg( m\frac{y^{n+1}-2y^{n}+y^{n-1}}{h^2}+b\beta_{ss}\frac{y^{n+1}-y^{n-1}}{2h}+c\beta_{ss}(\theta_1 y^{n+1}+\theta_2 y^{n}+\theta_3 y^{n-1})-$$

$$-b\beta_{cs}\frac{x^{n+1}-x^{n-1}}{2h}-c\beta_{cs}(\theta_1 x^{n+1}+\theta_2 x^{n}+\theta_3 x^{n-1})-F_y^n+\widetilde{Q}^n\Bigg)\Bigg(\dot v-\frac{y^{n+1}-y^{n-1}}{2h}\Bigg)+$$

$$+\varphi\widetilde{F}_y^n\left|\dot u-U\right|-\varphi\widetilde{F}_y^n\left|\frac{x^{n+1}-x^{n-1}}{2h}-U\right|\ge 0,\ \ \forall\{u,v\},\ n=2,3,...,\qquad(6)$$

where

$$\widetilde{F}_y^n=F_y\left(\theta_1 x^{n+1}+\theta_2 x^{n}+\theta_3 x^{n-1},\theta_1 y^{n+1}+\theta_2 y^{n}+\theta_3 y^{n-1}\right);\qquad(7)$$

$$\widetilde{Q}^n=\theta_1 Q^{n+1}+\theta_2 Q^{n}+\theta_3 Q^{n-1};\qquad(8)$$

$$\theta_1+\theta_2+\theta_3=1;\qquad(9)$$

$$x^0=x^1=0;\quad y^0=y^1=0.\qquad(10)$$

We introduce terms

$$d^{n+1}=\frac{x^{n+1}-x^{n-1}}{2h};\qquad(11)$$

$$e^{n+1}=\frac{y^{n+1}-y^{n-1}}{2h};\qquad(12)$$

$$\delta^{n+1}=\frac{x^{n+1}-x^{n}}{h};\qquad(13)$$

$$\gamma^{n+1}=\frac{y^{n+1}-y^{n}}{h}.\qquad(14)$$

Then, scheme (6) is as follows:

$$\Big(2m(d^{n+1}-\delta^n)/h+b\beta_{cc}d^{n+1}+c\beta_{cc}\left(2h\theta_1 d^{n+1}+\theta_2 x^n+(\theta_3+\theta_1)x^{n-1}\right)-$$
$$-b\beta_{cs}e^{n+1}-c\beta_{cs}\left(2h\theta_1 e^{n+1}+\theta_2 y^n+(\theta_3+\theta_1)y^{n-1}\right)\Big)\left(\dot u-d^{n+1}\right)+$$
$$+\Big(2m(e^{n+1}-\gamma^n)/h+b\beta_{ss}e^{n+1}+c\beta_{ss}\left(2h\theta_1 e^{n+1}+\theta_2 y^n+(\theta_3+\theta_1)y^{n-1}\right)\Big)-$$
$$-b\beta_{cc}d^{n+1}+c\beta_{cc}\left(2h\theta_1 d^{n+1}+\theta_2 x^n+(\theta_3+\theta_1)x^{n-1}\right)-F_y^n+\widetilde{Q}^n\Big)\left(\dot v-e^{n+1}\right)+$$
$$+\varphi F_y^{n+1}\left|\dot u-U\right|-\varphi F_y^{n+1}\left|d^{n+1}-U\right|\ge 0,\ \forall\{u,v\},\ n=2,3,...\qquad(15)$$

where

$$\widetilde{F}_y^{n+1}=F_y\left(2h\theta_1 d^{n+1}+\theta_2 x^n+(\theta_3-\theta_1)x^{n-1},2h\theta_1 e^{n+1}+\theta_2 y^n+(\theta_3-\theta_1)y^{n-1}\right).\quad(16)$$

To solve quasivariational inequality (15), iterative process (where $k$ is iteration index) is applied:

$$\Big(2m(d^{n+1}_{(k+1)}-\delta^n)/h+b\beta_{cc}d^{n+1}_{(k+1)}+c\beta_{cc}\big(2h\theta_1 d^{n+1}_{(k+1)}+\theta_2 x^n+(\theta_3+\theta_1)x^{n-1}\big)-$$
$$-b\beta_{cs}e^{n+1}_{(k+1)}-c\beta_{cs}\big(2h\theta_1 e^{n+1}_{(k+1)}+\theta_2 y^n+(\theta_3+\theta_1)y^{n-1}\big)\Big)\big(\dot{u}-d^{n+1}_{(k+1)}\big)+$$
$$+\Big(2m(e^{n+1}_{(k+1)}-\gamma^n)/h+b\beta_{ss}e^{n+1}_{(k+1)}+c\beta_{ss}\big(2h\theta_1 e^{n+1}_{(k+1)}+\theta_2 y^n+(\theta_3+\theta_1)y^{n-1}\big)-$$
$$-b\beta_{cc}d^{n+1}_{(k+1)}-c\beta_{cc}\big(2h\theta_1 d^{n+1}_{(k+1)}+\theta_2 x^n+(\theta_3+\theta_1)x^{n-1}\big)-\tilde{F}^n_{y,(k)}+\tilde{Q}^n\Big)\big(\dot{v}-e^{n+1}_{(k+1)}\big)+$$
$$+\varphi\tilde{F}^n_{y,(k)}\left|\dot{u}-U\right|-\varphi\tilde{F}^n_{y,(k)}\left|d^{n+1}_{(k+1)}-U\right|\ge 0,\ \forall\{u,v\},\ n=2,3,\ldots,\ k=1,2,\ldots, \quad (17)$$

where

$$\tilde{F}^n_{y,(k)}=F_y\big(2h\theta_1 d^{n+1}_{(k)}+\theta_2 x^n+(\theta_3-\theta_1)x^{n-1},\,2h\theta_1 e^{n+1}_{(k)}+\theta_2 y^n+(\theta_3-\theta_1)y^{n-1}\big). \quad 18)$$

It is easy to verify that if iterative process (17) converges then limit of sequence $\{d^{n+1}_{(k)},e^{n+1}_{(k)}\}$ is a solution for quasivariational inequality (15). Results of paper [8] show that iterative process (17) converges at any initial approximation, and peak limiting to friction value.

It is reasonable to use the following as initial approximation in iterative process (17)

$$d^{n+1}_{(0)}=d^n; \quad (19)$$

$$e^{n+1}_{(0)}=e^n. \quad (20)$$

Inequality (17) is variational inequality. Using results of paper [7], we can demonstrate that within each iteration a solution of variational inequality (17) is a solution of a problem of minimization of the two variables following function:

$$J_1(d,e)=\frac{1}{2}a_{11}d^2+a_{12}de+\frac{1}{2}a_{22}e^2-g_1 d-g_2 e+g_0\left|d-u\right|, \quad (21)$$

where

$$a_{11}=\frac{2m}{h}+b\beta_{cc}+2c\beta_{cc}h\theta_1; \quad (22)$$

$$a_{12}=-\beta_{cs}(b+2ch\theta_1); \quad (23)$$

$$a_{22}=\frac{2m}{h}+b\beta_{ss}+2c\beta_{ss}h\theta_1; \quad (24)$$

$$a_{21}=a_{12}; \quad (25)$$

$$g_1=2m\delta^n/h-c\beta_{cc}\big(\theta_2 x^n+(\theta_3+\theta_1)x^{n-1}\big)+$$
$$+c\beta_{cs}\big(\theta_2 y^n+(\theta_3+\theta_1)y^{n-1}\big); \quad (26)$$

$$g_2=2m\gamma^n/h-c\beta_{ss}\big(\theta_2 y^n+(\theta_3+\theta_1)y^{n-1}\big)+$$
$$+c\beta_{cs}\big(\theta_2 x^n+(\theta_3+\theta_1)x^{n-1}\big)+\tilde{F}^n_{y,(k)}-\tilde{Q}^n; \quad (27)$$

$$g_0 = \varphi \widetilde{F}^{n}_{y,(k)}. \tag{28}$$

Function $J_1(d,e)$ minimization can be expressed in an explicit form.

If $(g_1 a_{22} - g_2 a_{12} - g_0 a_{22})/a > U$, then

$$d = (g_1 a_{22} - g_2 a_{12} - g_0 a_{22})/a, \tag{29}$$

$$e = (g_2 a_{11} - g_1 a_{21} + g_0 a_{21})/a, \tag{30}$$

If $(g_1 a_{22} - g_2 a_{12} + g_0 a_{22})/a < U$, then

$$d = (g_1 a_{22} - g_2 a_{12} + g_0 a_{22})/a, \tag{31}$$

$$e = (g_2 a_{11} - g_1 a_{21} - g_0 a_{21})/a, \tag{32}$$

Otherwise

$$d = U, \tag{33}$$

$$e = (g_2 - a_{21} U)/a_{22}, \tag{34}$$

where $a = a_{11} a_{22} - a_{12} a_{21}$.

*Duplex Difference Schemes*

While using duplex schemes for timely integration of quasivariational inequality (1) we obtain:

$$\begin{aligned}
&\Bigg( m\frac{p^{n+1}-p^{n}}{h} + b\beta_{cc}\left(\theta_1 p^{n+1} + (1-\theta_1)p^{n}\right) + c\beta_{cc}\left(\theta_1 x^{n+1} + (1-\theta_1)x^{n}\right) - \\
&- b\beta_{cs}\left(\theta_1 q^{n+1} + (1-\theta_1)q^{n}\right) - c\beta_{cs}\left(\theta_1 y^{n+1} + (1-\theta_1)y^{n}\right)\Bigg)\left(s - p^{n+1}\right) + \\
&+ \Bigg( m\frac{q^{n+1}-q^{n}}{h} + b\beta_{ss}\left(\theta_1 q^{n+1} + (1-\theta_1)q^{n}\right) + c\beta_{ss}(\theta_1 y^{n+1} + (1-\theta_1)y^{n}) - \\
&- b\beta_{cs}\left(\theta_1 p^{n+1} + (1-\theta_1)p^{n}\right) - c\beta_{cs}\left(\theta_1 x^{n+1} + (1-\theta_1)x^{n}\right) - F_y^{n+\theta} + Q^{n+\theta}\Bigg)\left(w - q^{n+1}\right) + \\
&+ \varphi F_y^{n+\theta}\left|s - U\right| - \varphi F_y^{n+\theta}\left|p^{n+1} - U\right| \ge 0, \quad \forall\{s,w\}, \quad n = 2,3,...,
\end{aligned} \tag{35}$$

$$\frac{x^{n+1}-x^{n}}{h} = \theta_2 p^{n+1} + (1-\theta_2)p^{n}, \tag{36}$$

$$\frac{y^{n+1}-y^{n}}{h} = \theta_2 q^{n+1} + (1-\theta_2)q^{n}, \tag{37}$$

where $\{p^n, q^n\} = \{\dot{x}^n, \dot{y}^n\}$ are velocity components within $t^n$ time moment;

$$F_y^{n+\theta} = F_y\left(\theta_1 x^{n+1} + (1-\theta_1)x^{n}, \theta_1 y^{n+1} + (1-\theta_1)y^{n}\right); \tag{38}$$

$$Q^{n+\theta} = (\theta_1 Q^{n+1} + (1-\theta_1)Q^{n}). \tag{39}$$

(35)–(37) ratios are a system of quasivariational inequality and two algebraic equations. Initial conditions for (35) – (37) system will be selected as follows

$$x^0 = p^0 = 0; \tag{40}$$

$$y^0 = q^0 = 0. \tag{41}$$

Assume (36) – (37) equations as

$$x^{n+1} = x^n + h(\theta_2 p^{n+1} + (1-\theta_2)p^n), \tag{42}$$

$$y^{n+1} = y^n + h(\theta_2 q^{n+1} + (1-\theta_2)q^n) \tag{43}$$

and substitute (42)–(43) for (35). As a result, we obtain following quasivariational inequality

$$\begin{aligned}
&\Big( m\frac{p^{n+1}-p^n}{h} + \theta_1\beta_{cc}p^{n+1}(b+\theta_2 ch) + \beta_{cc}p^n((1-\theta_1)b + \theta_1(1-\theta_2)ch) + c\beta_{cc}x^n - \\
&- \theta_1\beta_{cs}q^{n+1}(b+\theta_2 ch) - \beta_{cs}q^n((1-\theta_1)b + \theta_1(1-\theta_2)ch) - c\beta_{cs}y^n\Big)(s - p^{n+1}) + \\
&+ \Big( m\frac{q^{n+1}-q^n}{h} + \theta_1\beta_{ss}q^{n+1}(b+\theta_2 ch) + \beta_{ss}q^n((1-\theta_1)b + \theta_1(1-\theta_2)ch) + c\beta_{ss}y^n - \\
&- \theta_1\beta_{cs}p^{n+1}(b+\theta_2 ch) - \beta_{cs}p^n((1-\theta_1)b + \theta_1(1-\theta_2)ch) - c\beta_{cs}x^n - \\
&- F_y^{n+\theta} + Q^{n+\theta}\Big)(w - q^{n+1}) + \\
&+ \varphi F_y^{n+\theta}|s-U| - \varphi F_y^{n+1}|p^{n+1}-U| \ge 0, \quad \forall\{s,w\}, \quad n = 2,3,\ldots,
\end{aligned} \tag{44}$$

where

$$F_y^{n+\theta} = F_y\left(x^n + \theta_1\theta_2 p^{n+1}h + \theta_1(1-\theta_2)p^n h,\ y^n + \theta_1\theta_2 q^{n+1}h + \theta_1(1-\theta_2)q^n h\right). \tag{45}$$

Iterative process (where $k$ is iteration index) is applied to solve the quasivariational inequality:

$$\begin{aligned}
&\Big( m\frac{p^{n+1}_{(k+1)}-p^n}{h} + \theta_1\beta_{cc}p^{n+1}_{(k+1)}(b+\theta_2 ch) + \beta_{cc}p^n((1-\theta_1)b + \theta_1(1-\theta_2)ch) + c\beta_{cc}x^n - \\
&- \theta_1\beta_{cs}q^{n+1}_{(k+1)}(b+\theta_2 ch) - \beta_{cs}q^n((1-\theta_1)b + \theta_1(1-\theta_2)ch) - c\beta_{cs}y^n\Big)(s - p^{n+1}_{(k+1)}) + \\
&+ \Big( m\frac{q^{n+1}_{(k+1)}-q^n}{h} + \theta_1\beta_{ss}q^{n+1}_{(k+1)}(b+\theta_2 ch) + \beta_{ss}q^n((1-\theta_1)b + \theta_1(1-\theta_2)ch) + c\beta_{ss}y^n - \\
&- \theta_1\beta_{cs}p^{n+1}_{(k+1)}(b+\theta_2 ch) - \beta_{cs}p^n((1-\theta_1)b + \theta_1(1-\theta_2)ch) - c\beta_{cs}x^n - \\
&- F_y^{n+\theta} + Q^{n+\theta}\Big)(w - q^{n+1}_{(k+1)}) + \\
&+ \varphi F^{n+\theta}_{y,(k)}|s-U| - \varphi F^{n+\theta}_{y,(k)}|p^{n+1}_{(k+1)}-U| \ge 0, \\
&\forall\{s,w\},\ n = 2,3,\ldots\ k = 1,2,\ldots,
\end{aligned} \tag{46}$$

where

$$F^{n+\theta}_{y,(k)} = F_y\left(x^n + \theta_1\theta_2 p^{n+1}_{(k)}h + \theta_1(1-\theta_2)p^n h,\ y^n + \theta_1\theta_2 q^{n+1}_{(k)}h + \theta_1(1-\theta_2)q^n h\right). \tag{47}$$

It is understood that if iterative process (46) converges then limit of sequence $\{p_{(k)}^{n+1}, q_{(k)}^{n+1}\}$ is the solution for quasivariational inequality (44). Results of paper [9] demonstrate that iterative process (46) converges at any initial approximation and peak limiting on friction coefficient.

Following value is taken as initial approximation in iterative process (46)

$$p_{(0)}^{n+1} = p^n, \tag{48}$$

$$q_{(0)}^{n+1} = q^n. \tag{49}$$

On determining values $\{p^{n+1}, q^{n+1}\}$ using iterative process (46), values $\{x^{n+1}, y^{n+1}\}$ are calculated with the help of (42) – (43) formulas.

Inequality (46) is variational inequality. Results of paper [7] demonstrate that each iteration of variational inequality (46) solution factors into solving the problem of two variables minimization following function:

$$J_1(p,q) = \frac{1}{2}a_{11}p^2 + a_{12}pq + \frac{1}{2}a_{22}q^2 - g_1 p - g_2 q + g_0|p-u|, \tag{50}$$

where

$$a_{11} = \frac{m}{h} + \theta_1\beta_{cc}(b + \theta_2 ch), \tag{51}$$

$$a_{12} = -\theta_1\beta_{cs}(b + \theta_2 ch), \tag{52}$$

$$a_{22} = \frac{m}{h} + \theta_1\beta_{ss}(b + \theta_2 ch), \tag{53}$$

$$a_{21} = a_{12}, \tag{54}$$

$$g_1 = m/h - \beta_{cc}p^n\big((1-\theta_1)b + \theta_1(1-\theta_2)ch\big) - c\beta_{cc}x^n + \\ + \beta_{cs}q^n\big((1-\theta_1)b + \theta_1(1-\theta_2)ch\big) + c\beta_{cs}y^n, \tag{55}$$

$$g_1 = m/h - \beta_{ss}p^n\big((1-\theta_1)b + \theta_1(1-\theta_2)ch\big) - c\beta_{ss}y^n + \\ + \beta_{cs}q^n\big((1-\theta_1)b + \theta_1(1-\theta_2)ch\big) + c\beta_{cs}x^n + F_y^{n+\theta} + Q^{n+\theta}, \tag{56}$$

$$g_0 = \varphi F_{y,(k)}^{n+\theta}. \tag{57}$$

(30) – (34) formulas can help solve minimization function $J_2(p,q)$ problem.

**Conclusions**. Solving dynamic problem with friction using developed computational algorithm help to obtain temporal series of a shoe motion $\{x^n\}$ and $\{y^n\}$. Research of the temporal series applies:

– Autocorrelation function to determine oscillation period;
– Spectrum analysis of motions, velocities, and accelerations;
– Phase diagrams in “motion-velocity” variables;

– Dependences of motion, velocity, and acceleration amplitudes on parameters of changes in dynamic system being considered; they are obtained using a method of parameter continuation.

The dynamic system is dissipative as it involves elastic and damping element. Hence in time the system motion becomes stable and periodic. Autocorrelation function determining for the temporal series makes it possible to define both its periodicity and its period duration if so.

The dynamic system is described with the help of non-linear dissipative non-autonomous system of ordinary differential equation. It is reasonable to divide motions of dissipative systems into the two classes: transient, nonstationary motions corresponding to a process of relaxation from initial two-boundary system of states, and a class of stable stationary motions which face trajectories completely belong to boundary systems.

It is determined that application of triplex differential schemes needs following values of balance coefficients:

$$\theta_1 = \theta_3 = 0.25\,, \quad \theta_2 = 1.0 - \theta_1 - \theta_3 = 0.5\,.$$

If differential schemes are duplex, it is reasonable to choose $\theta = 0{,}5$ balance coefficient.

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