MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE
NATIONAL TECHICAL UNIVERSITY
«DNIPRO POLITECHNIC»

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MODELS AND METHODS OF MAKING DECISIONS

A Coursebook

Dnipro
Dniprotech
2019

УДК 519.81(075.8)
U88

Рекомендовано вченою радою як навчальний посібник для студентів галузі знань 12 – інформаційні технології (протокол № 15 від 11 грудня 2018).

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U88 Us, S.A. Models and methods of making decisions: a coursebook / Svitlana A. Us, Larysa S. Koriashkina, Iryna I. Zuyenok; Ministry of Education and Science of Ukraine, Dnipro politechnic. – Dnipro : Dniprotech, 2019. – 304 p.

ISBN 978–966–350–708–8

This coursebook covers the materials within the content of "Decision-making theory" Syllabus, designed for undergraduate students specialized in Systems Analysis.

It encompasses main concepts and foundations of the theory of making decisions, typical methods of solving various problematic situations: choosing the best alternative from the given set, grouping and ranking alternatives and options, methods of making decisions for multicriteria problems, and making decisions in fuzzy conditions. Special attention is drawn to the applied aspects of making decisions and putting theory into practice when solving the tasks given.

The coursebook is intended for the students who have taken and passed the university course in higher mathematics, especially for those who are specialized in Systems Analysis. It is also useful for all who apply decision-making methods to solve real life problems and tasks.

Навчальний посібник охоплює матеріал, передбачений програмою дисципліни "Теорія прийняття рішень" для студентів , що навчаються за спеціальністю 124 – Системний аналіз.

Викладено математичні основи теорії прийняття рішень. Розглянуто типові методи виходу з проблемних ситуацій – вибір кращої альтернативи із заданої множини, групове впорядкування альтернатив, методи прийняття рішень за наявністю багатьох критеріїв та в умовах невизначеності. Основну увагу приділено прикладним аспектам теорії прийняття рішень.

Книгу розраховано на осіб, які опанували математику в межах курсу ЗВО, зокрема на студентів спеціальності «Системний аналіз». Вона може бути корисна всім, хто застосовує методи прийняття рішень до розв'язування практичних задач.

УДК 519.216/217(075.8)

ISBN 978–966–350–708–8

Присвячується 120-річчю
Національного технічного університету
«Дніпровська політехніка»

Передмова до видання англійською мовою

Увазі читача пропонується третє видання навчального посібника “Моделі та методи прийняття рішень”, орієнтоване на студентів – іноземних громадян закладів вищої освіти України, а також на тих, хто, опановуючи спеціальність «Системний аналіз», планує продовжити навчання за кордоном та/або брати участь у програмах академічного обміну. Посібник може бути корисним перекладачам науково-технічної інформації та викладачам, які читають дисципліни англійською мовою та ін.

Мета посібника – формування знань і розвиток умінь прийняття рішень на основі сучасних теорій та їх застосування на практиці у модельованих ситуаціях шляхом розв’язування наведених прикладних задач.

Мінімальна підготовка, необхідна для освоєння навчального матеріалу – це знання основ математичного аналізу і лінійної алгебри.

Велика увага у посібнику приділяється задачам багатокритерійної оптимізації, вивчаються питання, пов’язані із прийняттям рішень за наявності нечітких вихідних даних, в умовах ризику та невизначеності.

Виклад змісту дисципліни відбувається як у теоретичному, так і в практичному аспекті. З одного боку, матеріал націлено на закріплення базових теоретичних питань. Тут відпрацьовуються навички побудови математичних моделей задач, ідеї та алгоритми методів їх розв'язання. З іншого боку, навички грамотного практичного використання набутого матеріалу розвиваються при виконанні практичних завдань, перелік яких наведено у кожному розділі.

Навчальний посібник написано фаховою англійською мовою з огляду на рівень мовної підготовки цільової аудиторії – студентів ЗВО. Для полегшення розуміння теоретичного матеріалу автори пропонують англо-український глосарій термінів та іншу довідкову літературу.

Видання може бути корисним для різних категорій читачів. По-перше, це студенти вищих та інших закладів освіти, які вивчають дисципліни, пов'язані з сучасними інформаційними технологіями і комп'ютерним моделюванням. По-друге, це вже дипломовані фахівці, які бажають оцінити можливості комп'ютерної підтримки для вирішення внутрішніх проблем на своєму робочому місці. Нарешті, це сучасні керівники, які прагнуть застосувати у своїй роботі досягнення у сфері системного аналізу і математичного моделювання. Знання основних результатів і принципів теорії прийняття рішень та оптимізації дозволить не тільки особисто керуватися ними, а й ставити обґрунтовані завдання системному аналітику або відділу системного аналізу фірми. Крім того посібник може стати в пригоді перекладачам науково-технічної інформації та педагогам, які читають дисципліни англійською мовою.

CONTENTS

INTRODUCTION

One should not only hope that it is possible to make an error-free decision, on the contrary, one should agree in advance with the fact that every decision is doubtful, because it is an ordinary thing when having avoided one trouble, you fall into another. Wisdom consists of knowing how to distinguish the nature of trouble, and in choosing the lesser evil.

N. Machiavelli

The objects of modern mining: mines, collieries, ore quarries, concentrating factories and processing plants equipped with powerful mining equipment, are complicated complex enterprises. Managing such objects and planning their operations require a managing officer at any level to be able to make correct decisions quickly. In the conditions, when it is necessary to take into account many external factors, internal capacities and connections between the components of the objects under the investigation, the decisions made on the basis of personal experience and engineering intuition may be ineffective, if they haven't taken into account a number of contradictory conditions. Moreover, modern production is characterized by high cost consumption that raises sharply the losses from errors in forecasting and managing. This situation requires the use of a decision-making technology based on quantitative evaluation of options, which eliminates or reduces the values of subjective factors, while taking into account the influence of various inaccurate or undefined parameters. Only application of the systematic approach allows us to study decision-making problems in a situation where the choice of alternatives requires the analysis of complex information characterizing the real state of affairs.

Theory of decision-making is one of the most important sections of *the system analysis* which encompasses a set of methods based on the use of computer information technology. It is focused on the study of complex systems: technical, economic, social, environmental, software ones etc. The usual result of such studies is the choice of a certain alternative: a plan of the company or corporation development, the design parameters, project management strategies etc.

The essence of decision-making as a process is the internal relatively stable logical basis which determines the meaning, role and place of a particular managing decision for organization functioning and/or its development. The essence of decision-making is usually manifested through the implementation of the variety of external connections and actions, accompanying the process. Following this, one can determine a subject of study of the decision-making theory.

The main aim of any management decision is to provide a coordinating (regulatory) impact on the entire management system when making its tasks and achieving the goals of the organization. The main tasks, which make up the content and sequence of actions of decision-makers in carrying out their immediate duties, are creating the appropriate information base; defining its limitations and decision-making criteria; organising management system activities. *Decision-making is a creative and responsible process.* It aims to determine the tactics of further actions in a particular area of production of goods or services in accordance with the circumstances; to outline the functions of the structural units in the system of the organization's activities, the order of their interaction, provision and management.

For making decisions in-time, it is necessary to have a management system which ensures the implementation of complex systematic activities of decision-makers and to organize the work of the firm (corporation) on the scientific basis by using effective methods and automated control systems. At the same time, the quality of the decisions made largely depends on the coherence of the team-work, the inherent organizational culture, the relationships between executives and performers, and on the successful use of decision-making support systems. The substantiated practical recommendations made on the basis of the decision-making theory with the account of objective laws and achievements of the related sciences can be of great use, especially with dealing with these issues.

The subject of the study of the decision-making theory is the laws (regularities) of the actions of the people who make decisions, organizational forms, technologies and methods of this activity, the principles of management and organization of work, the nature and content of the decisions.

The object of the decision-making theory is referred to as the systematic activity of a head or leader and their teams in the process of developing, adopting and implementing decisions.

Consequently, *the decision-making theory* is the sum of knowledge of the preparation, acceptance and implementation of management decisions, the laws and principles, organizational forms, methods and technologies for providing this process within the subject of entity.

The decision-making theory like any scientific theory performs both cognitive and predictive functions. The cognitive function is fulfilled by revealing the nature of the decision-making processes, the regularities and principles they are subjected to, explaining their theoretical foundations at different historical stages, highlighting the main properties and interrelations of the subject of the research, substantiating technology and decision-making system. The predictive function includes identifying trends in the further development of processes and decision-making systems, ways of choosing the organizational forms and methods of the management object operations.

The main tasks of the decision-making theory are following:

– to study and generalize the experiencies of finding solutions in the certain situations, under conditions of uncertainty and risk in particular;

– to detect and study objective laws in decision-making processes, form on their bases the principles of organizing the activities of decision-makers, organizational forms involved in them, and development methods and technologies;

– to develop practical recommendations for decision-making to be used by linear managers and for operations of the CEOs who manage them in the real life situations as well as recommendations for the use of technical facilities and automated control and management systems;

– to develop the research methods for studying problems of a decision-making system, principles and methods for evaluating their effectiveness as well as measures to improve the decision-makers' activities.

Currently, the development of the decision-making theory is significantly influenced by other sciences, including methodology, in particular methodology of thinking, theory of management, cybernetics, psychology, sociology and political science. In this aspect, fundamental significance is also acquired by natural sciences: biology, psychophysiology. But, of course, mathematics plays a decisive role, in particular the methods of quantitative evaluation of options in making decisions and predicting the development of situations, while developing the most rational solution.

This coursebook contains the detailed description of mathematical models and methods which are used for formalization and substantive justification of a decision.

In dependence on what concepts are taken as the key ones to formalize the problem there are different approaches to decision-making.

For example, if to assume that decision-making is the choice of the most successful alternative from the set of available ones, then the task is described by a pair (Ω, C), where Ω is the set of possible alternatives, C is the principle of optimality. This approach corresponds to the situation when the external environment does not influence the decision-making result.

In the statistical approach, the situation of the decision-making is described by a triple: $(\mathit{\Phi}, \Theta, F)$, where Φ is the set of possible decisions of the control management body, Θ – the set of states of the environment, F is the estimated functional.

In this edition, decision-making is considered primarily within the first approach.

The basis of the coursebook is made of the materials of *Theory of Decision-making* Course of Lectures delivered to the students specialized in "System Analysis and Management". The authors of the course aim is to develop students' ability to formalize the tasks of planning, organizing and managing in the field of mining, to construct economic and mathematical models of production situations and to make decisions based on the calculation of optimal variables of changed in the terms of certainty and uncertainty.

To be a success in learning the coursebook materials, a student needs to have knowledge in mathematical analysis, linear algebra, matrix theory, mathematical programming and optimization methods. The students who have already mastered standard finite-level optimization methods can apply some of their specific evaluations and possible alternative approaches to solving practical problems.

The coursebook is divided into six sections. Each section contains theoretical part and self-study part, wherein the latter is focused on self-assessment of understanding theoretical issues and putting theory into practice by solving proposed tasks.

The first section deals with general approaches to formulating and solving decision-making tasks (DMT). There one can find the classification of these tasks, the types of uncertainties that may arise in the decision-making process and the ways of their formalization.

The second section describes the basics of the theory of choosing options from a given set of alternatives. Here the basic concepts of the theory of binary ratios are determined, and their use in decision-making tasks is considered. In addition, the function of choice and some elements of the theory of usefulness are described.

The third section focuses on the characteristics of decision-making tasks of many criteria, in particular multi-criteria optimization tasks. The methods of solving these tasks and problems as well as methods for taking into account the criteria of priorities and normalization are given there.

In the fourth and fifth sections, the content of decision-making tasks in the fuzzy conditions, including tasks of fuzzy mathematical programming, is determined. Their classification is given, and the methods of solution are described (section 4), and the selection problems are considered on the basis of fuzzy preferences (section 5).

The sixth section includes an outline of the theory of choice of the options from a given set of alternatives in the presence of various types of uncertainty. In particular, the problem of choice in conditions of "environment" uncertainty (decision-making under conditions of risk, complete uncertainty, in game situations of choice) is considered.

To easify understanding of the coursebook materials the authors provide English-Ukrainian, Ukrainian – English Glossaries of the main terms that make the coursebook useful for translators.

Although references to the various sources are limited in the text, the list of references given in the end of the coursebook provides a complete list of sources used, which may be useful as an additional source for Ukrainian students when solving practical tasks, and the list of sources in English to be used while self-study.

SECTION 1

DECISION-MAKING TASKS AND THEIR CLASSIFICATION

By the end of this section you will:

- become familiarized with problems for making decisions, their typical tasks and the existing approaches to their solving;
- have studied basic concepts of the theory of decision-making.

1.1 Examples of decision-making tasks and their classification

Consider some of the tasks, the purpose of which is to make decisions.

- The position of the Chief Engineer has been resigned at an enterprise. The director should appoint a new manager, choosing one of the possible applicants for this position, taking into account the data about their education, professionalism, work experience, authority in a team, age, communication skills, health and other factors, each of which has a different significance in the selection procedure.

- Some mines and quarries should ship the mined coal to consumers at several locations. It is known, how much coal is produced at each of the enterprises and the needs of each of the consumption points. It is necessary to organize the delivery of the raw material in such a way which minimizes the costs of mileage and freight.

- Assume that at the quarry there is a possibility to use several types of transport for goods transportation. It is necessary to divide the amount of coal extracted in each lava by dumps, by types of transport and reloading points, i.e. to determine how much coal and what type of transport should be used to transport coal from each lava to each point of reception (dump, transshipment point).

- When designing a quarry, it is necessary to choose the kind of quarry transport which will provide minimal expenses on transportation processes provided by using automobile and rail road transport or their combinations. In addition, it is necessary to take into account the amount of rock mass that must be transported, the cost and transportation volumes of each mode of the transport, and location of the transshipment point when using their combination.

- The mine needs to load a certain amount of ore during the given period, providing a minimum number of impurities in the raw material, meeting the requirements for raw material conditions, and uniform loading of the equipment and minimal transportation costs.

This task includes the following sub-tasks: planning of mineral resource production aimed to meet the necessary requirements to its conditions and volume; planning loading of the equipment: optimizing transportation procedure(s).

- Developing the construction plan of an open pit aimed at putting it into operation as soon as possible.

- Choosing the location of a concentrating plant serving a group of mines and quarries, taking into account minimization of transportation costs and the social and environmental requirements.

- Determining the optimal plan for using various resources.

- Planning and managing mining works in averaging mode. The task of this type can be formulated as follows:

Minerals that are extracted in separate sites of a mining enterprise have different content of useful and harmful components, while the processing plants (concentrators, smelters, power plants) impose strict requirements to the quality of raw materials. Therefore, it is necessary to plan the mining process on each of the sites in such a way that the overall quality of the products meets the requirements of consumers with the manufacturing process being the most effective.

- Allocate tasks among the faces taking into account the plan of production, capacity of the transport routes, the power of lava, necessary repair and preparation works.

- Determine the plan for supplying the central coal processing plant with coal from different mines, the implementation of which will provide the minimum costs for transportation, smooth functioning of the plant and the proper quality of raw material.

- Develop an optimal plan for ore extraction from several mines, at one concentrating plant.

- Assume that there are several types of equipment that can be used at different production sites. The number of equipment of each type and its capacity (in each site of its own) are known. So, it is necessary to allocate the equipment in such a way that the total time spent on the task is minimal.

- Mining machine factories produce a variety of equipment that is used in coal industry associations for extracting of coal of various grades. The production of each type of product requires a certain amount of resources (financial, labour, raw material, material), and consequently each kind of such products has its own cost and price. In the conditions where the maximum possible and the minimum required quantity of equipment of each type is known, it is necessary to make the best plan for its release.

- Transportation of goods at the enterprises of mining industry takes a significant place in the country's cargo turnover. It is necessary to plan the transportation and/or construction of new mining enterprises in order to minimize the shipping costs.

- There are several faces in the quarry. The volumes of work on each of them are known. There are also a few dumps the reception capacity of which is also known. It is necessary to plan the transportation of rock mass from the faces to the dumps in such a way that transport costs are minimal.

- It is necessary to choose the routes of coal's transportation (each route is characterized by the following parameters: length, load, safety level, availability of maintenance, filling stations etc.) and distribute transport units between routes, taking into account the existing fleet of equipment, the possibility of attracting additional funds, the need to complete the order within a specified period etc. (*tasks of optimal organization of transportation*).

- Choose the equipment for carrying out works, taking into account its cost, productivity, ecological requirements, qualifications of personnel etc. (*tasks of choice*)

- Develop a mining company project.

- Determine the optimal conditions of the technological processes in mining enterprise.

- Develop an optimal mining plan for the company.

- *Distribution* tasks may be as following:

– distribute excavators among working places (faces, categories of rocks) in order to minimize freight costs;
– distribute cars within the routes;
– distribute employees by type of work;
– distribute machine tools among workers;
– distribute motor vehicles by excavators.

All of these tasks like many others are characterized by the fact that making decisions in them is a conscious choice one of the possible alternatives (depending on the specific content, they are called strategies, plans, options) done on the basis of a certain principle (criterion) of optimality.

This choice is made by a *decision maker* (DM). The role of a decision-maker can be performed by individual people or groups of people, who have the right to make a choice and choose and who are responsible for the consequences. It may be a supervisor, a dispatcher, a manager of the shift, workshop or a head of the department, CEO, a Director General or a Board of Directors. Based on the available data, including mathematical calculations and studies, DM chooses the final solution option within their competences.

Consequently, any decision-making process can be characterized by the following elements:

1. The person who makes a decision.
2. A set of variables which values are chosen by a DM: alternatives, strategies, plans, guides.
3. A set of variables, the values of which depend on the decision made: results, output variables of the decision-making situation.
4. A set of variables which values DM doesn't regulate (parameters and the environment).
5. The time interval during which decisions are being made.
6. Mathematical model of the task for decision-making on a problem, which is a set of correlation between parameters, guiding/managing actions and output variables.
7. Restrictions which describe the requirements caused by the decision-making situation in relation to the output variables of the task and the guiding/managing actions.
8. The target function or the criterion of optimality with the help of which the quality of the chosen solution is evaluated.

Each of these elements can be characterized by a different degree of uncertainty. Depending on this, different classes of decision-making tasks can be formed.

If the parameters and external perturbations (i. e. the environmental impact) remain unchanged over time, the mathematical model will be static. Otherwise, the model of the decision-making situation will be dynamic. The description of the static model can be presented as a graph, table, functional dependence or an algorithm for calculating the output variables. Dynamic models are described using various classes of differential or difference equations.

When the external perturbations are non-random, we have a *deterministic* decision-making model, but if they are random, then we obtain a *stochastic* model. Under this condition, the output variables will also be random, and their distribution will depend on the distribution of external perturbations.

In the case where the set of possible alternatives and the optimality criterion are fully defined, the problem of decision-making is reduced to the *optimization* problem.

When a situation requires several criteria to be considered, it is described with the help of the multicriteria optimization problem.

If the set of alternatives is defined, the optimality criterion is unknown, but relations of preference defined on the set of alternatives are known, then we are dealing with *the task of making choice*. This is a fairly common situation since it is not always possible to quantify each alternative, but it can often be determined which of these two alternatives has advantages over the other or several other pairs. At the same time, when some or all of the elements of the task have uncertainty such as "*fuzzy*", then we have *decision-making tasks in fuzzy conditions*. In particular, these are the tasks of fuzzy mathematical programming, the tasks of choice in fuzzy conditions etc.

Depending on the class of the task, an appropriate approach to its solution is chosen. These may be methods of optimization, linear or non-linear programming, statistical methods, analytical or numerical methods for solving equations of different classes.

Thus, in the decision-making process, there are situations which have one or another degree of uncertainty That is why the quality of the decision depends on the completeness of the consideration of all the factors influencing the consequences of the decision made. These factors are often subjective, and this true both for DM and for the decision-making process itself. Moreover, a managing body does not always have all the information that is necessary for its substantiated actions.

So, the main difficulties which arise in the decision-making process can be distinguished, namely:

1. The presence of a large number of criteria that are not always consistent, i.e. correlated with each other. For example, during the process of designing a new device for the aircraft the requirements to minimizing its mass are specified as well as maximum reliability and minimum cost. These criteria are contradictory. Therefore, the task of finding a compromise solution that takes into account all the requirements arises.

2. A high degree of uncertainty caused by the lacks in the information taken for a well-founded decision-making.

Such situations require a special mathematical apparatus for their description that would provide the possibilities of taking into account such uncertainties.

The methods of probability theory, game theory, statistical decisions, fuzzy sets, or qualitative methods of system analysis may be appropriate and used. Scheme of the decision-making tasks classification by the number of their criteria, dependence on time, random factors as well as data of the corresponding mathematical apparatus is given in Figure 1.1. The scheme is presented in the manual [4].

1.2 Uncertainty in the decision-making tasks

Uncertainty in decision-making is caused by the lack of reliability and the amount of information on the basis of which DM makes his/her choice.

Here is a classification of uncertainty made by types and causes of its occurrence.

1. Principal uncertainty caused by the impossibility to obtain information in principle. For example, at this level of development of scientific knowledge.

2. Uncertainty caused by the total number of objects or elements of the system. For example, when their number exceeds 10^9.

3. Uncertainty caused by lack of information or its uncertainty due to the technical, social or other reasons.

4. Uncertainty generated by the too high or inaccessible price necessary for establishing certainty.

5. The uncertainty created by a decision maker due to their incompetence, lack of experience and knowledge of the factors that influence the process.

6. Uncertainty as a consequence of constraints in the decision-making system (time and space constraints of parameters that characterize decision-making factors).

7. Uncertainty caused by non-antagonistic behavior of the opponent, which has an influence on the decision-making process.

Another classification of types of uncertainty involves:

- obscurity;
- incompleteness;
- insufficiency;
- inadequacy;
- underdetermination.

Schematic correlations between these types is shown in Figure 1.2.

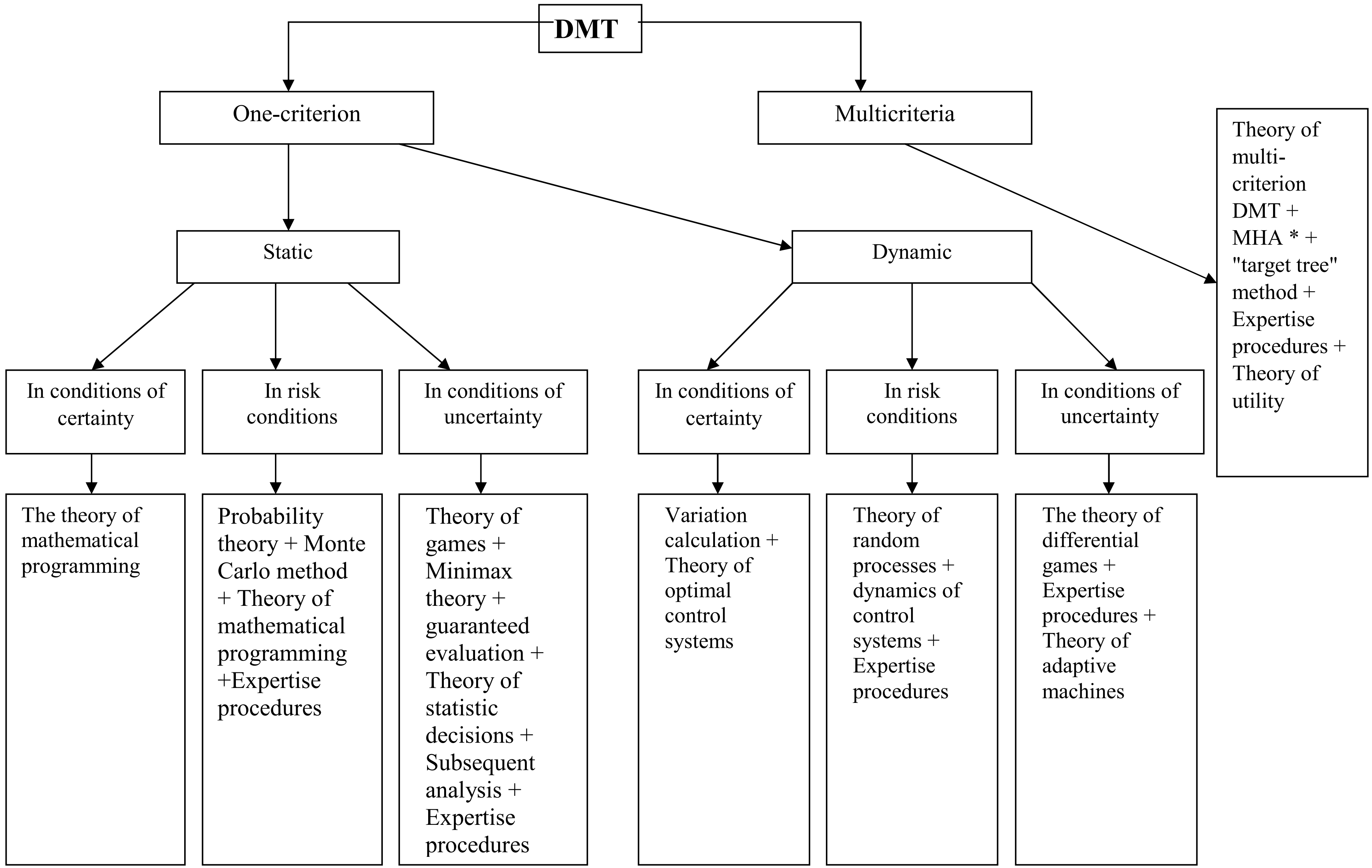


Fig.1.1. Classification of decision-making problems and mathematical apparatus that is used for solving them

* MHA – method of hierarchy analysis

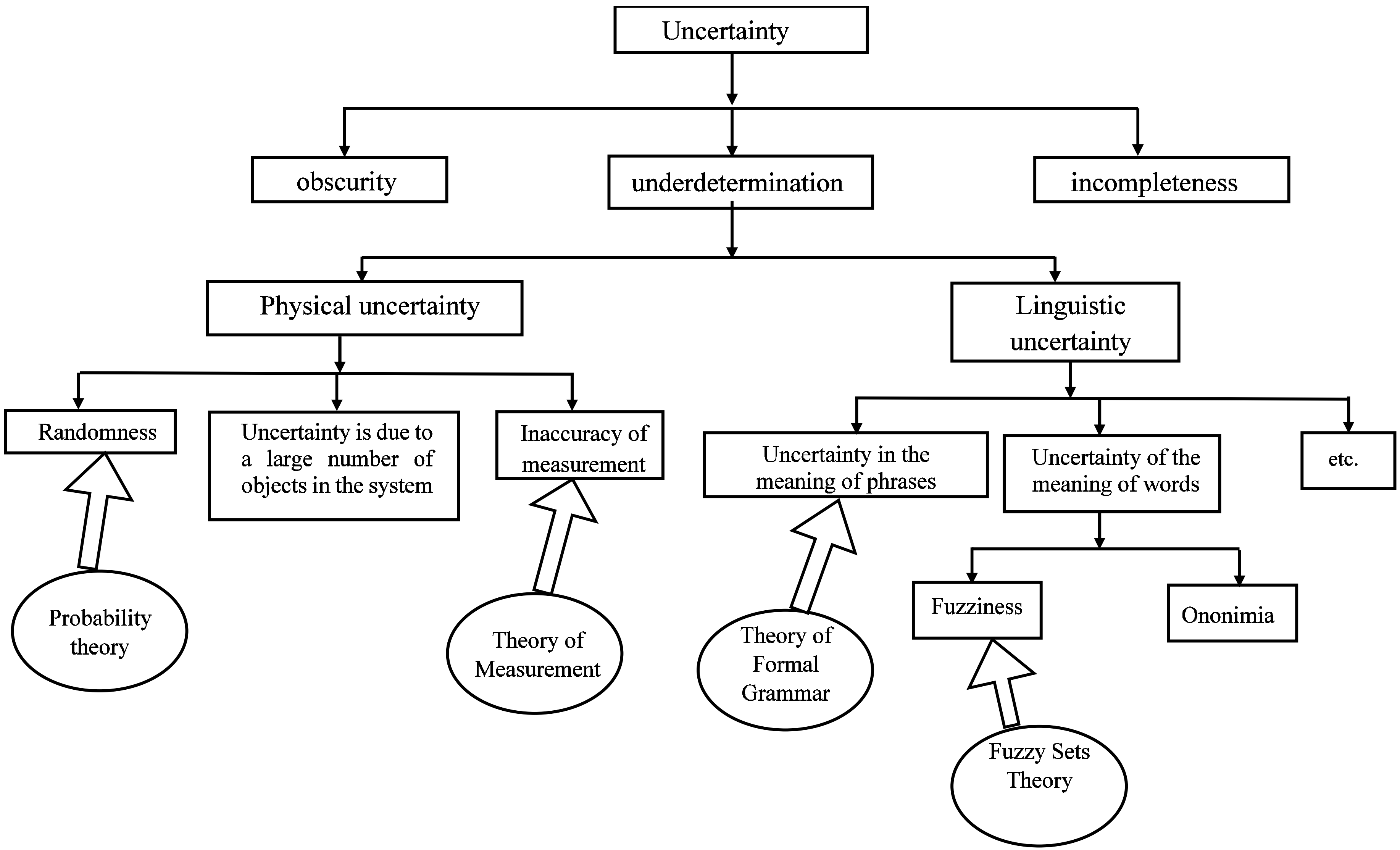


Fig. 1.2. The correlation between different types of uncertainty

1.3 Theoretical-game approach to decision-making

Decision-making theory encompasses several approaches, depending on the elements considered to be the main when analyzing the process of decision-making.

According to the theoretical game concept, decision-making is a choice of the better alternative made from the set of available ones.

Consequently, the integral components of such model will be the set of alternatives and the description of the considerations of a decision maker. It should be noted that in real-life tasks alternatives have many properties that influence the solution.

Let some property of alternatives from a plural Ω be described by a number, i.e. there is a mapping $\varphi : \Omega \to R_1$. Then, such property is called *a criterion* and the number $\varphi(x)$ is an estimate of the alternative x by the criterion φ.

In the decision-making tasks, the criteria serve to express the "intensity" of the essential properties (signs) of solutions. They are divided into quantitative and qualitative by their nature. Each criterion is associated with a set of permissible transformations Φ and then we say that this criterion has a scale of type Φ.

The criteria which have a scale that is not less perfect, than the interval (i. e, their permissible additions are multiplication to a positive number and the addition of an arbitrary number r) are called *quantitative*.

The criteria with a sequence scale (all monotonically increasing functions are assigned to them) are called *qualitative*. The meaning of a qualitative criterion makes sense to compare with others only by the ratio of "more", "less", "equal".

The simultaneous consideration of individual properties of alternatives may be a complex process. Thus, groups of properties that aggregate in the form of aspects are identified.

Aspect is a composite property of alternatives which simultaneously takes into account all the properties belonging to a certain group. In a particular case, an aspect may be a criterion.

Example 1.1. The transport agency needs to transport a specified volume of cargo. A dispatcher must determine the route of transportation.

In this task, alternatives are different routes. The dispatcher must consider the following properties: extent (length), load capacity, safety level, all the expenses related to transportation process, specifity of maintenance etc.

The term "maintenance" includes the number and location of service stations, their capacity, load capacity and the period of repair works. Thus, this characteristic is an aspect that aggregates all the listed properties.

The length of the route is measured in kilometers, i.e. expressed by the number and therefore it can be considered as a criterion.

In the general case, the value of the criterion depends on two groups of factors

– controlled (guided) factors which depend on DM and represent their strategy (choice);

– uncontrolled factors, the ones which is DM cannot influence, are referred to as parameters of the decision-making task. They may be determined, stochastic or indefinite (see above).

The value of managed factors is usually limited to a number of natural conditions, for example, constraints of the resources. These conditions are the basis for the formation of *constraints* for the task for decision-making.

If during the process of decision-making, it is necessary to take into account several properties of alternatives, a problem of multi-criteria choice appears.

Assume all the properties k_1, k_2 …, k_m of the alternatives taken into account when solving a problem, are the criteria. So, put the k_j criterion in accordance with the j axis of the space R_m (j = 1, … , m) and represent a plural Ω in this space, putting in correspondence with each alternative $x \in \Omega$ the point: $\varphi(x) = (\varphi_1(x), \dots, \varphi_m(x))$, space R_m, where φ_j is the estimate for the criterion k_j (j = 1, … , m).

The *criterion space* is called the space R_m, which point coordinates represent estimates for the corresponding criteria.

Thus, in the multicriteria task of comparing alternatives in favour of the advantages is carried out by using the set of numeric functions $\varphi_1(x), \dots, \varphi_m(x)$ given on the set Ω.

For each criterion φ_j on a numerical line (the R_j axis) a subset Y_j from which it acquires values is described. Practically, the set Y_j is determined according to the meaning of this criterion.

The criteria of φ_j (.) are called partial or local. They form a vector criterion:
$\varphi(x) = (\varphi_1(x), \dots, \varphi_m(x))$.

We will assume that every alternative x is completely described by the corresponding vector estimate, i.e. the vector $\varphi(x)$. That is why the choice of the optimal solution is reduced to the definition of the optimal estimate from the set of the achievable: $Y = Y(x) = \left\{ y \in R_m \middle| y = \varphi(x), x \in X \right\}$.

In real problems, a set Y cannot often be constructed; therefore, a wider set $Y' \in R_m$ is considered, the vectors of which can be given a certain content.

In the situation, where the available information is not enough to quantify estimate of each alternative, but there is a possibility in respect to some (or all) pairs of alternatives to determine which one is better, for their comparing the apparatus of binary relations is used.

Conclusions

The decision-making problem is one of the key issues in human activities. The main difficulties encountered during the decision-making process are the presence of a large number of mutually uncoordinated criteria and a high degree of uncertainty rooted in the lack of information necessary for a well-founded decision-making.

There are different approaches for decision-making: game theory, optimization, statistics, and others, according to what elements are considered to be the principal in the analysis of this process.

The methods, used in decision-making depend on the nature of the task, the available information and the chosen approach to its solution.

SELF-STUDY

Questions for assessment and self-assessment

1. Give some examples of decision-making tasks.
2. Give a definition of the criterion, aspect, principle of optimality, limitations in the theory of decision-making.
3. What problems arise in the decision-making process?
4. What factors determine the quality of the decision-making process?
5. What are the characteristics of the classification of decision-making tasks?
6. What mathematical apparatus is used for solving decision-making problems?
7. What types of uncertainty exist?
8. What approaches are used in solving non-classical decision-making problems?
9. What provisions include theoretical-game approach to decision-making?

Hands-on practice

Task A

Describe the set of feasible alternatives, options, constraints and criteria for the following tasks:

1. The head of the company has to decide which program for accounting from those on the market (for example, 1C, "Parus", C2, "Accountant-3", a custom-made program) must be purchased, taking into account the following factors: cost, information security, possibility and flexibility of setting, resource requirements etc.

2. It is necessary to determine among the clients of the company the most promising person(s) for signing long-term contracts.

3. CEO (managers) of the factory examines promising projects for the development of an enterprise, each of which requires some resources and takes into account certain factors (funds, raw materials, terms of realization, personnel potential, etc.). You need to choose one or more projects to be implemented.

4. Taking into account the information of existing fixed assets of an enterprise, its personnel potential, raw materials, infrastructure as well as the information about partners, competitors, market conditions, influence of state regulation, financial support, it is necessary to make a choice of the direction of the enterprise activities: development of basic production, re-profiling, increase in exports, the possibility of entering or refusing from the markets etc.

5. Determine the minerals which are expediently to extract in the region (recommended variants: coal, iron ore, phosphates, calyx), taking into account the efficiency and cost of their extraction.

6. Every day passenger and fast trains go from point A to point B. Information about the existing fleet of different types of railcar, from which trains can be completed, and the number of passengers that can be carried by each type of a railcar is given in Table 1.1. Determine the optimal number of fast and passenger trains which provide the maximum number of passengers transported.

Table 1.1

Type of railcar	Fleet of railcars	Train		Number of passengers
		fast	passengers	
Luggage	12	1	1	–
Post	18	1	–	–
Solid	89	5	8	58
Compartment	79	6	4	40
Soft	35	4	2	32

7. Three types of coal-plow machine I, II, and III which are able to perform three types of works A, B, and C can be used in the operations of a quarry. The resources of working time of each longwall, their capacity during performing various tasks and the cost of one hour of work (in UAH) is given in Table 1.2. Determine the optimal load of combines, which provides the maximum total volume of work performed and their minimum cost.

Table 1.2

Type of coal-plow machine	Productivity, m3 / h			Specific cost, UAH per year			Time resource
	A	*B*	*C*	*A*	*B*	*C*	
I	30	20	40	2	4	2	400
II	20	30	50	3	2	5	300
III	60	40	20	5	3	6	280

8. At mine "Dobropilska" there are three extraction sites. The coal extracted on each of them has a different sulfur content, different indexes of moisture and ash content (see Table 1.3). For each of the sites, the values of the maximum possible and the minimum required amount of extraction are known as well as the cost of extraction of one ton of raw materials (Table 1.3). Due to the characteristics of the coal produced at each site, it is necessary to draw up a plan of work in such way, that the extraction costs are minimal, its volume is maximal, and all the requirements of the consumers to the quality of raw materials are fulfilled (see in Table 1.4).

Table 1.3

Characteristics of coal, % and performance indicators of the site	Site number		
	1	2	3
Ash content	49	37	23
Moisture	7	8	10
Sulfur content	1.8	2.1	3
Costs, UAH	1184.210	1381.777	1083.515
Maximum volume of extraction, ths. tons	1650	1090	1270
Minimal amount of raw material extraction, ths. tons	1200	600	530

Table 1.4

Quality of coal	Ash content, %	Moisture, %	Sulfur content, %
Operational	39.5	–	–
Average	–	8.2	2.16
No more	47.4	9.8	2.6

9. The airline company for the organization of passenger transportation between the center C and four cities M1, M2, M3, M4 has three groups of airplanes. The first group consists of 10 four-engined airplanes, the second – of 25 twin-engined

planes of the new model and the third – of 40 two-engined aircraft of the old model. Data on the number of passengers that can be transported by one aircraft of this type for each route within one month and associated operating costs per 1 aircraft (ths. UAH) is given in Table 1.5. The number of passengers to be transported for each route within a month is respectively 40, 50, 40, and 30 thousand of people, and the cost of one ticket is 200, 150, 180 and 300 UAH respectively. It is necessary to distribute aircrafts among the routes, proceeding from the condition of achieving the maximum airlines revenue and the maximum number of the transported passengers.

Table 1.5

Type of airplane	Number of passengers / operating costs, ths. UAH			
	C - M1	C - M2	C - M3	C - M4
I	320/1.2	300/0.8	190/1.5	250/1.6
II	200/1.4	250/1.5	170/2.0	260/2.9
III	225/1.0	300/1.1	200/1.8	320/1.7

10. The processing plant receives 4 types of coal in the following quantities: 400, 250, 350 and 100 thousand tons. As a result of mixing these four components in different proportions, three grades of concentrate are formed: A (1: 1: 1: 1), B (3: 1: 2: 1) and C (2: 2: 1: 3). The cost of 1 thousand tons of concentrate is 120, 100 and 150 UAH, respectively. Determine the optimal production plan to achieve its maximum total cost and maximum quantity.

11. The company received two batches of plywood, with the volume of the first batch – 400, and the second – 250 sheets. 4 details of type 1, 3 details of type 2 and 2 details of type 3 are made from these sets. One sheet of plywood from the first batch can be cut in three ways: R1, R2, R3; the plywood from the second batch can be cut in four ways: R1, R2, R3, R4. The data on the number of details of each type, which can be cut from one sheet in one way or another, is given in the Table 1.6. It is necessary to cut the existing material in such way so to ensure the production of the maximum number of sets.

Table 1.6

Type of details	Number of details, шт.						
	The firts batch			The second batch			
	R1	R2	R3	R1	R2	R3	R4
1st	0	6	9	6	5	4	0
2nd	5	3	4	5	3	2	6
3rd	12	14	0	7	4	5	7

12. Five technological processes (T1, T2, T3, T4, T5) can be involved in the enterprise operations and the number of product units produced with the use of each of them per unit time which is equal to 300, 260, 320, 400 and 450 pcs, respectively. The following factors are taken into account in the technological process: amount of raw materials, energy consumption, wage costs and overhead costs. Their values when working for a unit of time in applying to various technologies are summarized in Table 1.7.

Task: Identify a production program that maximizes product output.

Table 1.7

Production resources	Costs for different technologies					Resource limit
	T1	T2	T3	T4	T5	
Raw materials, ton	15	20	15	14	18	2000
Electricity, kW	0.2	0.3	0.15	0.25	0.3	300
Overhead costs, UAH	4	5	6	3	2	1000
Salary, UAH	6	3	4	6	3	1600

13. The mechanical factory uses turning, milling and planing lathes for manufacturing parts of I, II and III types. The processing of the parts of each type can be carried out in three different technological ways T1, T2 and T3. Table 1.8 gives the time limits for the processing of a part on the corresponding machine by each of technological methods as well as the time resources (in machine-hours) for each group of machine tools. Profit from the sale of each type of the product is respectively 22, 18 and 30 UAH. Make the optimal plan for boosting production capacity, which ensures maximum profit with the minimum use of lathes.

Table 1.8

Type of machine tool	Standards for processing parts, year									Time resource
	I			II			III			
	*T*1	*T*2	*T*3	*T*1	*T*2	*T*3	*T*1	*T*2	*T*3	
Lathe	1	0.9	1.1	1.2	1.5	–	0.9	–	–	200
Milling	0.8	0.8	1.3	0.9	1.1	1.3	1.1	0.8	–	400
Planing	–	0.7	0.7	0.7	–	1.3	1.3	0.6	–	300

14. For producing an alloy from lead, zinc and tin of the certain percentage content raw materials in the form of five alloys of the same metals having different composition and cost are used (see Table 1.9). Determine the amount of alloy of each type to be taken to produce an alloy containing tin - from 40 to 60% and zinc - from 20 to 30% with a minimum cost.

Table 1.9

Type of an alloy	Metal content, %			Specific cost, UAH / kg
	Lead	Zinc	Tin	
I	25	30	45	8
II	*10*	*80*	*10*	*17*
III	30	30	40	10
IV	40	25	35	12
V	10	70	20	15

15. Solve the task 14, taking into account additional conditions necessary to make the maximum amount of an alloy, whith the available reserves of alloys I – V are 20, 25, 15, 30, 20 kg, respectively.

16. Details *A*, *B*, *C* can be processed on three machines: I, II, III. The rates of time spent on the processing a corresponding part with the machine, the cost per hour of the machine work and and the maximum time of its operation are given Table 1.10. Assuming that any part can be processed on any of the machines, determine the optimal production program according to one of the following criteria: the maximum of commodity products; minimum production cost.

Table 1.10

Machine tools	Rate of processing time			Expenses per hour of work, UAH	Operating time of the machine, h
	A	*B*	*C*		
I	0,3	0,1	0,2	30	50
II	0,5	0,2	0,4	20	60
III	0,4	0,5	0,3	15	40

17. Using data of Table 1.10 and assuming that every detail is consistently processed on all of the machines, make a production program that ensures the maximum output of marketable products at a minimal cost.

Task B

For tasks 6 – 17 make the mathematical models of linear programming (models can have one or several criteria).

SECTION 2

CHOICE PROBLEMS

By the end of this section you will:

- become aware of the apparatus of binary relations and its use in decision-making systems;
- have studied methods of making decisions on the basis of the given relations of preference, functions of choice and utility functions;
- have developed skills of implying these methods in practice.

2.1 Concepts of binary relation

The simplest situation where it is possible to make a selection or reasonable choice of several objects occurs when one "quality criterion" is given that allows to compare any of two objects, to specify exactly which one of them is the best, and to select the one (or more) for which this criterion reaches the maximum value. However, in most real situations, it is difficult to determine this one criterion and sometimes impossible at all. But considering some pairs of objects, it is possible to name the best of them. In such cases, it is said that these two objects are in *binary relation*. This concept allows us to formalize operations of pairwise comparison of alternatives therefore, it is widely used in the theory of decision-making.

Let's consider some of the statements that express the interrelationships between objects.

1. Tatiana is older than Igor.
2. Companies A and B are loss-making.
3. Kyiv is located more to the South than Moscow.
4. Ivan is Peter’s brother.
5. Iron is heavier than water.

As we see, these statements describe the relations of different types:

For example, statements 2 and 4 mean that two objects are assigned to the same class; statements 1, 3, 5 reflect the order of objects in the system. In addition, all five examples clearly identify the names of the objects and the names of relations. It is easy to see that when one object is replaced by another one, the following situations are possible:

1) the relation will be fulfilled, i.e. true, again (Kyiv is located more to the South than Murmansk);

2) the relation will not be fulfilled, i.e. wrong (Kyiv is located more to the South than Odessa);

3) the relation will not make any sense (Iron is located more to the South than water).

So, we can talk about the relation only when we are able to allocate the set of objects on which it is defined.

Mathematically, the definition of a relation can be formulated as follows:

Definition 2.1. The *relation* R in the set Ω is referred to as a subset of the Cartesian product $\Omega \times \Omega$, i.e. $R \subset \Omega^2$.

Setting a subset R in set $\Omega \times \Omega$ determines which pairs of elements are in relation R.

The relation R given in set Ω is denoted by the (R, Ω). Hereinafter: $x\,R\,y$ or $(x, y) \in R$, means that the elements x and y of the set Ω are in relation R.

2.2 Ways of representation relations

In order to determine a relation (R, Ω), it is necessary to specify all pairs of elements $(x, y) \in \Omega \times \Omega$ that are included in set R. In addition to the complete list of all pairs, there are three ways of setting relations: using a matrix, a graph and cuts. The first two methods are used to determine the relation on finite sets; the relation with cuts can also be applied to infinite sets.

The detailed description of the named methods of assigning relations is given in the subsections given below.

2.2.1 Assigning relation using a matrix

Let set Ω be composed of n elements, R is the binary relation given in this set. Number the elements of the set Ω by integers from 1 to n. In order to establish a relation, construct a square table of $n \times n$ size. Its i-th line corresponds to element x_i of set Ω, the j-th column is an element x_j of set Ω. At the intersection of the i-th row and the j-th column, put 1, if element x_i is in relation R to element x_j, and zero in other cases, namely:

$$a_{ij}(R) = \begin{cases} 1, & x_i R x_j, \\ 0 & \text{in other case.} \end{cases}$$

Example 2.1. Let $X = \{1, 2, ..., 5\}$, R – the relation "is greater than" in set X. Then it can be described as a matrix in the following way:

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

2.2.2 Assigning a relation using a graph

In order to assign a relation by means of a graph, we put the vertices of the graph $x_1,...,x_n$ (of any numbering) in a mutually unambiguous match the elements of the finite set Ω which defines the relation.

Make an arc from vertex x_i to x_j, only in the case if the element x_i is in relation to element R. When $i = j$ the arc (x_i, x_j) becomes a loop at vertex x_i.

Example 2.2. Assign the relation from Example 2.1 using a graph (see Figure 2.1).

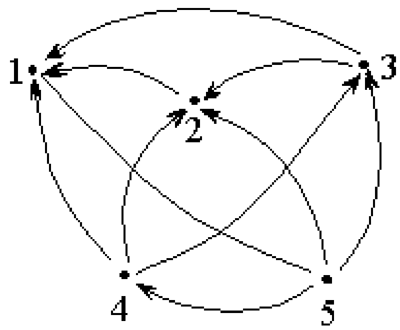


Fig. 2.1. Assigning relation ‘more” in the set: $X = \{1, 2, 3, 4, 5\}$, using a graph

So, when any directed graph G with n vertices is given and the numbering in set Ω is chosen, where the set consists of n elements, that is in such a way, a certain relation is given on this set: $R = R(G)$, namely, the statement $x_i \, R \, x_j$ will be valid only when there is an arc (x_i, x_j) in graph G. Consequently, the graph acts as a geometric representation of the relation.

2.2.3 Setting relations using cuts

Consider relation R on set Ω.

Definition 2.2. *The upper cut* of the relation (R,Ω) in element x denoted by $R^+(x)$, is referred to as a set of elements $y \in \Omega$ for which the condition: $(y,x) \in R$ is being satisfied, that is

$$R^+(x) = \{y \in \Omega | (y,x) \in R\}. \tag{2.1}$$

Definition 2.3. *The lower cut* $R^-(x)$ of the relation (R,Ω) in element x is referred to as a set $y \in \Omega$ of elements for which $(x,y) \in R$, namely:

$$R^-(x) = \{y \in \Omega | (x,y) \in R\}. \tag{2.2}$$

Consequently, the upper cut (set R^+) is a set of all such elements y that are in relation to R with a fixed element x $(y\ R\ x)$. The lower cut (set R^-) is a set of all elements y such that the fixed element x is in relation to $R(x\ R\ y)$.

Thus, in order to assign a relation with cuts, it is necessary to describe all upper or all lower cuts. In other words, relation R will be defined if for each element $x \in \Omega$ a set is given, or a set is given for each element $x \in \Omega$.

Example 2.3. Let a set be defined: $\Omega = \{1, 2, 3, ..., 10\}$. Relation R means "to be a divisor", i.e. $x R y$, if x is a divisor of y. This relation can be assigned in the following ways:

using the upper cuts:

$R^+(1) = \{1\}$, $R^+(2) = \{1; 2\}$, $R^+(3) = \{1; 3\}$, $R^+(4) = \{1; 2; 4\}$, $R^+(5) = \{1; 5\}, R^+(6) = \{1; 2; 3; 6\}$ $R^+(7) = \{1; 7\}$, $R^+(8) = \{1; 2; 4; 8\}$, $R^+(9) = \{1; 3; 9\}$, $R^+(10) = \{1; 2; 5; 10\}$;

using the lower cuts:

$R^-(1) = \{1; 2, ..., 10\}$, $R^-(2) = \{2; 4, ..., 10\}$, $R^-(3) = \{3; 6; 9\}$, $R^-(4) = \{4; 8\}$, $R^-(5) = \{5; 10\}$, $R^-(6) = \{6\}$, $R^-(7) = \{7\}$, $R^-(8) = \{8\}$, $R^-(9) = \{9\}$, $R^-(10) = \{10\}$·

Consider the relations of a special form and the methods described above for their assignment.

A relation is referred to a called *empty or null* (denoted by $\varnothing$) if it is not executed for any pair $(x, y) \subset \Omega\times\Omega$.

For the empty/null relation the following statements are true:

1. In matrix A ($\varnothing$) the values $a_{i,j}(\varnothing) = 0$ for all values of i, j.
2. The graph G ($\varnothing$) has no arcs.
3. $R^{+}(x) = R^{-}(x) = \varnothing$ for each element $x \in \Omega$.

The relation is called *full* or *universal* (denoted as U) if it is executed for all pairs $(x, y) \subset \Omega\times\Omega$. For the full relation the following features are true.

1. In matrix $A(U)$ the values $a_{i,j}(U) = 1$ for all values of i, j.
2. In the graph $G(U)$ arches connect any pair of vertices.
3. Cuts $R^{+}(x) = R^{-}(x) = \Omega$ for all elements is $x \in \Omega$.

The relation is called *diagonal* or *identity* relation, or the relation of equality (denoted by E) when it is true for all pairs $(x, y) \subset \Omega\times\Omega$, which consist of equal elements. That is $x\, E\, y$, if and only if $x = y$. For the diagonal relation E, the following statements can be used.

1. In matrix A (E)

$$a_{ij}\left(E\right)=\begin{cases}1, & if \quad i=j,\\ 0 & \text{in other case.}\end{cases}$$

2. In graph G (E) the loops are only at the vertices; the other arcs are absent.
3. The cuts $R^{+}(x) = R^{-}(x) = x$ for all elements $x \in \Omega$.

The relation is called *anti-diagonal* (denoted $\overline{E}$) when it is true for all pairs $(x, y) \subset \Omega\times\Omega$, which consist of non- equal elements. The following features are valid for the relation $\overline{E}$:

1. In matrix $A(E)$

$$a_{ij}\left(\overline{E}\right)=\begin{cases}1, & if \quad i\neq j,\\ 0 & \text{in other case.}\end{cases}$$

2. There are all arcs (x_i, x_j) in graph G ($\overline{E}$) if $i \neq j$ (there are no loops only at the vertices y).
3. The cuts $R^{+}(x) = R^{-}(x) = \Omega \setminus \{x\}$ for all elements $x \in \Omega$.

2.3 Operations on binary relations

Definition 2.4. The relation R_1 is *included* in relation R_2 (written as $R_1 \le R_2$) when the set of pairs, for which the relation R_1 is performed, is included in the set of pairs, for which R_2 is true.

We will say that the relation R_1 is *strictly included* in R_2 $(R_1 < R_2)$, if $R_1 \le R_2$ and $R_1 \ne R_2$. Equality of relations is realized in the same way as the equality of sets.

For a matrix assignment of relations, the following rule will apply: if $R_1 \le R_2$, then $a_{ij}(R_1) \le a_{ij}(R_2)$, $i, j = \overline{1,n}$.

Example 2.3. R_1 - the relation "$\le$" in the set of real numbers, R_2 – the relation "<" is in the same set, then $R_2 \le R_1$.

Definition 2.5. Relation $\overline{R}$ is called an *complementary* relation to R, if only it holds those pairs of elements for which the relation R is not true.

Obviously, that

$$\overline{R} = \Omega^2 \setminus R. \tag{2.3}$$

Given this in a matrix record $a_{ij}(\overline{R}) = 1 - a_{ij}(R)$, $i, j = \overline{1,n}$.

In graph $G(\overline{R})$ there are those and only those arcs that are not in graph $G(R)$.

The following statements are true for the cuts of relation $\overline{R}$:

$$\overline{R}^+(x) = \Omega \setminus R^+(x),$$

$$\overline{R}^-(x) = \Omega \setminus R^-(x).$$

Example 2.4. Let R be the relation "$\ge$" given in the set of real numbers, then $\overline{R}$ – the relation "<" given on the same set.

Definition 2.6. The *intersection* of relations R_1 and R_2 (written as $R_1 \cap R_2$) is the relation determined by the intersection of the corresponding subsets of set Ω^2.

In matrix record, this means that

$$a_{ij}(R_1 \cap R_2) = \min\{a_{ij}(R_1), a_{ij}(R_2)\}, \ i, j = \overline{1,n}.$$

Definition 2.7. The *union* of relations R_1 and R_2 (denoted by $R_1 \cup R_2$) is the relation obtained by merging or combining the corresponding subsets of set Ω^2.

In matrix record, this can be applied as follows:

$$a_{ij}\left(R_1 \cup R_2\right) = \max\left\{a_{ij}\left(R_1\right), a_{ij}\left(R_2\right)\right\}, \quad i, j = \overline{1, n}.$$

Definition 2.8. A *converse* relation R is the relation R^{-1} that satisfies the following condition:

$$x\, R^{-1} y \Leftrightarrow y\, R\; x. \tag{2.4}$$

For matrices of relations R and R^{-1}, the following formula will be true:

$$a_{ij}\left(R^{-1}\right) = a_{ji}\left(R\right).$$

Example 2.6. Let relation R be given in the set $X = \{x_1,\ x_2,\ x_3, x_4, x_5\}$ by such a matrix:

$$R = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

The task is to construct the correspondent converse relation and possible additions.

Solution

According to Definition 2.5, relation R can be specified by the following matrix:

$$\overline{R} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

We construct a converse relation in accordance with Definition 2.8. Hence,

$$R^{-1} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Definition 2.9. The *product* (or *composition*) of relations R_1 and R_2 (denoted by $R_1 \cdot R_2$) is called the relation which is constructed according to the following rule:

$x\left(R_1 \cdot R_2\right) y$ when there is an element $z \in \Omega$ that satisfies the conditions $x\, R_1\, z$ and $z\, R_2\, y$.

Example 2.7. Consider relations R_1 and R_2, given in the set of real numbers. Moreover, R_1 is the relation "less than", R_2 is the relation "greater than". A pair of numbers $(x, y) \subset R_1 \cdot R_2$ when a z number exists, for which the following requirements are met: $x < z$ and $z > y$. Obviously, this condition is fulfilled for all numbers *x*, *y*, and therefore, $R_1 \cdot R_2$ is a complete relation, i.e. such a relation where all the elements of the given set are connected).

Example 2.8. The set $X = \{x_1,\, x_2,\, x_3, x_4, x_5\}$ has two relations R_1 and R_2, namely:

$$R_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Determine their composition.

Solution

According to Definition 2.9, $x\left(R_1 \cdot R_2\right)y$, if there is an element $z \in \Omega$ that satisfies the conditions $x\, R_1\, z$ and $z\, R_2\, y$. In matrix record, this means that

$$a_{ij}\left(R_1 \cdot R_2\right) = \max_{k=\overline{1,n}} \min\{a_{ik}\left(R_1\right), a_{kj}\left(R_2\right)\},$$

where n – is the order of the matrix.

In other words, the composition of relations is calculated as the maximinum product of the corresponding matrices.

Then we get the following result:

$$R_1 \cdot R_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Definition 2.10. The relation $\left(R_1, \Omega_1\right)$ is called the restriction of the relation $\left(R, \Omega\right)$ to set Ω_1, if $\Omega_1 \subset \Omega$ and $R_1 = R \cap \Omega_1 \times \Omega_1$. The restriction of the relation $\left(R, \Omega\right)$ to the set Ω_1 is also called as the relation R on the set Ω_1.

Example 2.9. The relation «>» in the set of positive integers is a restriction of the relation «>» on the set of real numbers.

2.4 Features and properties of relation

Definition 2.11. The relation R is called *reflexive* , if $x\,R\,x$ for any element $x \in \Omega$.

For example, the relation of "to be alike", "not to be older", "less or equal" is reflexive; "To be a brother", "to be older", "more" – not reflexive.

Units are placed on the main diagonal of a reflexive matrix that the matrix element $a_{ij} = 1$, if $i = j$.

A graph of a reflexive relation obligatory has a loop at the vertices. For the upper and lower cuts, the following statements are true: $x \in R^{+}(x)$, $x \in R^{-}(x)$ for all elements $x \in \Omega$.

Definition 2.12. Relation R is called *irreflexive*, when the statement $x\,R\,y$ means that $x \neq y$ for $\forall x \in \Omega$.

In a matrix of irreflexive relation, the elements of the main diagonal are zero, that is $a_{ij} = 0$, if $i = j$.

A graph of an irreflexive relation does not have loops at the vertices, and the upper and lower cuts satisfy the following conditions: $x \notin R^{+}(x)$, $x \notin R^{-}(x)$ for all elements $x \in \Omega$.

Irreflexive relations will be the relations of "more", "less", "to be older".

Definition 2.13. Relation R is called *symmetric* if $R = R^{-1}$ $(x\,R\,y \Rightarrow y\,R\,x)$.

The matrix of symmetric relation is symmetric that is $a_{ij} = a_{ji}$ for all values of i, j. In the graph of this relation, all arcs are paired, and the upper and lower cuts coincide for all elements $x \in \Omega$, i.e. $R^{+}(x) = R^{-}(x)\ \forall x \in \Omega$.

The symmetric relations are relations of equality: "to be similar", "to study in one group" are symmetric.

Definition 2.14. Relation R is called *asymmetric* if $R \cap R^{-1} = \varnothing$, i.e. from two expressions $x\,R\,y$ and $y\,R\,x$ at least one does not correspond to reality.

In the matrix of asymmetric relation $a_{ij} \wedge a_{ji} = 0$ for all values i, j, that is from two symmetric elements a_{ij} and a_{ji} at least one necessarily equals 0.

For example, asymmetric relations are the relations of "more" and "less".

Note that irreflectivity is an obligatory condition for asymmetry.

Definition 2.15. The relation R is called *antisymmetric* if the statements $x\,R\,y$ and $y\,R\,x$ can be valid simultaneously only if $x = y$.

The matrix has an antisymmetric relation $a_{ij} \wedge a_{ji} = 0$ when $i \neq j$.

Examples of antisymmetric relations will be "more or equal", "is not more", "is not worse."

Definition 2.16. The relation R is called *transitive* if $R^2 \leq R$ i.e. when $x\,R\,y$ follows from the assertions that $R^2 \leq R$ and $z\,R\,y$.

Transitive relations are the relations: "more or equal", "less", "to be older", "study in one group".

Condition: $R^2 \leq R$,provides a convenient way to check the transitivity of the relation in the case when the relation is given using the matrix. To do this, it is necessary to calculate a matrix of the relation R^2 , i.e. to bring the square of the matrix of the output relation, and verify the condition. If $a_{ij}\left(R^2\right) \leq a_{ij}(R)$ for all values of $i, j,$ then the relation is transitive. If this condition is violated at least for one pair of indices i, j, then the relation will not be transitive.

Definition 2.17. The relation R is called *acyclic* , if $R^k \cap R^{-1} = \varnothing$, i.e.the conditions $x\,R\,z_1$, $z_1\,R\,z_2$, ..., $z_{k-1}\,R\,y$ leads to $x \neq y$.

This means that the graph of this relation does not contain cycles.

Definition 2.18. The relation R is called a *negative transitive* if its complement $\overline{R}$ is transitive.

Definition 2.19. The relation R is called *highly transitive* if it is simultaneously transitive and negative transitive.

Properties of acyclicity and transitivity play a special role in the theory of decision-making, because they express the natural interrelations between objects. Indeed, if object x is in some sense not worse than object y, and the object y is in the same sense not worse than the z object, then it is natural to expect that the object x will not be worse than the object z (transitivity), and in any case the object z is not better than the object x (acyclicity).

Example 2.10. Determine the properties of this relation:

$$R = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Solution

This relation is a reflexive one since its matrix contains only units on the main diagonal. It won’t be symmetric since there are elements that are not equal to symmetric elements, e.g. elements a_{12} and a_{21}. Since element $a_{13} = a_{31}$, then the relation will not be asymmetric or asymmetric.

To verify its transitivity, multiply this relation by itself:

$$R^2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Since $R^2 \not\subset R$, therefore, the outgoing relation is not transitive.

2.5 Relations of equivalence, order, dominance and preference

Definition 2.20. A relation *R* is the relation of equivalence or *equivalnce*, if it is reflexive, symmetric and transitive. Let's denote it with R_e or the ~ character.

Examples of equivalence relations will be such as:

– "to take one course", i.e. “to study at one course”, "to study in one group" are given in the set of students of the faculty;

– "have the same remainder when dividing by 3"- in the set of positive integers;

– the relation of similarity in the set of triangles, and others.

Equivalence is characterised by distribution of the elements into classes. In the first example, these are courses or groups of the faculty students, in the second there are sets of numbers having the same remainder when dividing by 3, in the third, the set of similar triangles. Consequently, the problem of equivalence in a set is closely related to its partition in non-intersect subsets. Let’s consider this feature of equivalence in detail.

Let some partition of the set is Ω, i.e. its subsets are known as Ω_1, Ω_2, … Ω_N, which satisfy the condition: $\Omega = \bigcup_{i=1}^{N} \Omega_i$ and also $\Omega_i \cap \Omega_j = \varnothing$, when $i \neq j$, $i, j, = 1, 2, \ldots N$. We introduce the relation *R* in the set Ω as follows: $x\,R\,y$ if and only there is a set Ω_i corresponding to the following conditions: $x \in \Omega_i$ and $y \in \Omega_i$.

Task. Prove that the relation, which is introduced in this way, is equivalence.

As we see, the equivalence problem in a certain set Ω is equivalent to the division of this set into classes of elements equivalent to each other. Conversely, any partition of the set defines the equivalent of it.

Definition 2.21. The relation of the *non-strict order* «$\leq$» is the relation that has the properties of reflexivity, antisymmetry and transitivity.

Definition 2.22. The relation of the *strict order* «<» is called the relation which has the properties of irreflexivity, asymmetry and transitivity.

If the relation «$\leq$» is given in a set Ω, i.e. there is some non-strict order, then it can be placed in correspondence with the strict order of «<» which is defined by the following rule: $x < y$ if and only if $x \leq y$ and $x \neq y$. Conversely, if «<» is the relation of the strict order given in the set Ω, then it can be matched to the relation «$\leq$» in this way only if $x < y$ or $x = y$. Consequently, by a non-strict order we can determine the corresponding strict order and vice versa.

Assume that in a certain set the relation of order (for all or some pairs of its elements) is given, then it is said that a *partial order* is given in this set.

Partial order in a set Ω is called *a linear order*, if for any elements one of three statements: $x < y$, $x = y$ or $x >$ *is valid,* i.e. we can compare any two elements of the set Ω).

Definition 2.23. The relation of *dominance* is a relation that has the properties of antireflexivity and asymmetry.

We will say that element *x* dominates element *y* if *x* in some sense is better than *y*.

Thus, the relation of strict order is a separate case of the relation of dominance, for which transitivity is also a characteristic. In general sense, with dominance both transitivity and acyclicity may not take place.

Definition 2.24. Two elements can be compared by a relation *R*, when $x\,R\,y$ or $y\,R\,x$. In other cases, the elements are *not comparable*.

If *R* is a complete relation in a set Ω, then any two elements of this set can be compared.

Let us consider what orders can be set in *m*-dimensional space E_m:

1. $a \geq b$ if and only if $a_i \geq b_i$, $i = 1,2,\ldots m$;

2. $a \geq b$ if and only if $a_i \geq b_i$, $i = 1,2,\ldots m$, and $a \neq b$;

3. $a > b$ if and only if $a_i > b_i$, $i = 1,2,\ldots m$;

4. $a \mathrel{\overline{>}} b$ if and only if $a = b$ або $a_i > b_i$ or at least for one value $i \in \{1,2,\ldots m\}$;

5. $a = b$ if and only if $a_i = b_i$, $i = 1,2,\ldots m$.

Relation 1 is a partial order, it is reflexive, antisymmetric and transitive.

Relations 2 and 3 are strict partial orders. They are irreflexive, asymmetric and transitive.

Relation 4 is reflexive, but it will not be either symmetric or transitive.

The interrelation between these relations is schematically illustrated in Figure 2.2.

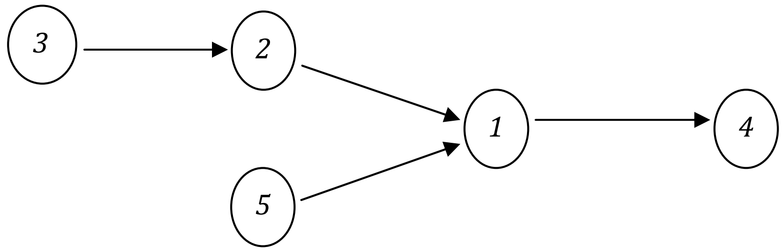


Fig. 2.2. Scheme of interrelation between relations in space E_m

In order to describe preferences, the following binary relations are usually used in the set of alternatives Ω: strict preferences, indifference and non-preferential preferences.

The relation of the strict preferences R^S means that one object (strictly) prevails over another that is, one object is better than another.

The relation of indifference R^I means that the preferences of objects are the same, and when limiting the choice by these two objects, it does not matter which one is selected.

The relation of non-strict preferences means that one object is not less important than the other, that is one object is not worse than another.

Assume that with the help of DM or experts, the relation of non-strict preference R in the set of admissible alternatives X was determined.

This means that one of the following situations is possible with respect to any pair of alternatives:

1) the object x is not worse than the object y, that is $x > y$, in other words $(x, y) \in R$;

2) the object y is not worse than the object x, i.e. $y \geq x$ or або $(y, x) \in R$;

3) objects x and y are not comparable with each other, ie $(x, y) \notin R$ and $(y, x) \notin R$.

This information allows us to narrow the class of alternatives of rational choice by including only those alternatives over which no other alternative of set X dominates.

To explain this concept, we will determine the strict preference R^S and the relation of indifference I which correspond to preference relation R.

We will say that the alternative x is strictly better than the alternative y (has a strict superiority over the alternative y) if at the same time $x \geq y$ and $y \neq x$ that is

$$(x, y) \in R \ \text{ i } \ (y, x) \notin R .$$

The set of all such pairs is called *the relation of the strict preference* R^S in the set *X*.

It's easy to make sure that this relation meets the following properties:

1) irreflexivity;

2) asymmetry.

For a more compact relation R^S we use the definition of the relation R^{-1} converse to *R*, we take into account, that $(x,y) \in R^{-1} \Leftrightarrow (y,x) \in R$, namely.

Then the relation of the strict preference can be written in the following way:

$$R^S = R \setminus R^{-1}.$$

The relation of *indifference*, corresponding to the preference relation *R* can be defined as follows: $(x,y) \in R^I$ if only either of the following conditions: $x \geq y$ or $y \geq x$ is not fulfilled or both occur at the same time: $x \geq y$ and $y \geq x$. In other words, $(x,y) \in R^I$ when the information we have is insufficient for a reasonable choice between alternatives *x* and *y*.

Mathematically, the relation R^I can be written by the following formula:

$$R^I = \left[(X \times X) \setminus \left(R \bigcup R^{-1}\right)\right] \cup \left(R \bigcap R^{-1}\right).$$

It's easy to make sure that the more information about a real situation or process, the closer the relation of indifference is revealed.

The introduced relations will be placed in Table. 2.1.

Table. 2.1

№	Relation name	Property/Feature					
		Reflexivity	irreflexivity	Symmetry	Asymmetry	Antisymmetry	Transitivity
1	Preference	+					
2	Strict preference	+			+		
3	Similarity	+		+			
4	Equivalence	+		+			+
5	Strict order		+		+		+
6	Non-strict order	+				+	+
7	Domination		+		+		

Example 2.11. Let the set be $X=\{x_1,x_2,x_3,x_4\}$, the relation of non-strict preference is given in this set, i.e.

$$R=\begin{pmatrix}1&1&0&1\\0&1&0&1\\0&1&1&0\\1&0&1&1\end{pmatrix}.$$

Construct the corresponding equivalence relation, strict preference, indifference.

Solving

According to the definition $R^e=R\bigcap R^{-1}$, we firstly construct a converse relation, namely

$$R^{-1}=\begin{pmatrix}1&0&0&1\\1&1&1&0\\0&0&1&1\\1&1&0&1\end{pmatrix}.$$

Now we are finding the relation of equivalence

$$R^e=R\bigcap R^{-1}=\begin{pmatrix}1&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&1\end{pmatrix}.$$

As it can be seen from this matrix, the elements x_1,x_4 are equivalent.

Now, in accordance with the definition, we will find the relation R^S in the following way:

$$R^S=R\setminus R^{-1}=\begin{pmatrix}0&1&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{pmatrix}.$$

This means that the element x_1 strictly prevails the element x_2, the element x_2 in its turn is more important than x_4, the element x_3 prevails x_2, and x_4 is better than x_3 respectively.

The relation of indifference can be found by the following formula:

$$R^I = [(X \times X) \setminus (R \bigcup R^{-1})] \cup (R \bigcap R^{-1}).$$

The matrix of this relation takes the following form:

$$R^I = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

This relation means that among the elements $\{x_1, x_3\}$, $\{x_1, x_4\}$, $\{x_3, x_4\}$ one can choose anyone, that the information is not enough in order to make a reasonable choice between the elements of each pair.

When $(x, y) \in R^S$, then we will say that the alternative x dominates the alternative y $(x > y)$.

Definition 2.25. We call the alternative $x \in X$ *non-dominant* for the set X by the relation *R*, if $(y, x) \notin R^S$, $\forall y \in X$. That is, if the alternative x is non-dominant, then in the set X there is no alternative that would dominate the alternative x.

In the above shown example, an alternative x_1 has been found to be non-dominant.

If some alternatives are in a certain sense non-dominant, then their choice in decision-making tasks is appropriate to be considered rational within the available information.

Thus, information in the form of a preference relation allows us to reduce the class of rational solutions in the set X to the set of non-dominated alternatives, which has this form:

$$X^{N.} = \left\{ x \mid x \in X, \ (y, x) \in R \setminus R^{-1}, \forall x \in X \right\}.$$

2.6 Concepts of *R*-optimality, of the best, the worst, the maximum and the minimum elements

The discussed above material was intended to give a formal description of the pair comparison of alternatives, which is a prerequisite for the selection of the best element (or several best ones) from the whole set of alternatives X. Now we formalize the concept of "the best" by using the apparatus of binary relations.

Definition 2.26. Element x^* of the set X will be called *the best* with respect to the relation *R* if $x^* \ R \ x$ is true for each element $x \in X$.

Definition 2.27. An element $x_* \in X$ will be called *the worst* with respect to the relation *R*, if $x\, R\, x_*$ for all $x \in X$.

It's easy to make sure that the best and the worst elements do not always exist. In particular, they will not exist when the relation is not complete, as in the following example.

Example 2.12. Consider the set: $B = \{a,b,c\}$, and the relation *R* in it, which is given as following: $R = \{(a,a),(a,b),(b,b),(b,c)\}$. Determine the best and the worst elements in the set *B*, if such exists.

Let's depict the relation described using the graph (see Figure 2.3)

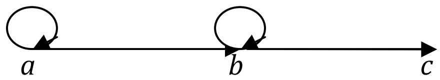


Fig. 2.3. Graph of relation *R* (for *Example 2.12*)

As we see, this relation does not have the best and the worst elements, because the elements *a* and *c* are incomparable.

Let us introduce the concept of the maximal element.

Definition 2.28. An element $x_{\max}$ is called *maximal* with respect to the relation R^S in the set *X* if for any element $x \in X$ there is a statement $x_{\max}\, R^S\, x$ or $x_{\max}$ incomparable with *x*.

In other words, there is no element (alternative) $x \in X$ that would be better than the alternative $x_{\max}$.

The set of maximal relations *R* of elements of the set *X* is denoted by $\max_R X$.

Definition 2.29. An element $x_{\min}$ is called *minimal* relatively to R^S in the set *X*, if for all $x \in X$ or $x\, R^S x_{\min}$, or *x* will be incomparable with it. S,o there is no element $x \in X$ that would be worse than $x_{\min}$; There is no element *x* over which the element $x_{\min}$ dominates.

In the example above, the element *a* will be maximal, and the element c will be the minimum.

The set of minimum values of *R* of elements of the set *X* is denoted by $\min_R X$.

Note that when the best elements exist, they will be maximal, and the opposite situation will not be fair.

Consequently, if one has to choose the best alternative in a certain sense, then it will be natural to choose it from a set of maximum (nondominated) alternatives.

Example 2.13. Let a relation R be in the form of graph G (Figure 2.4). Find the best, worst, maximal and minimal elements.

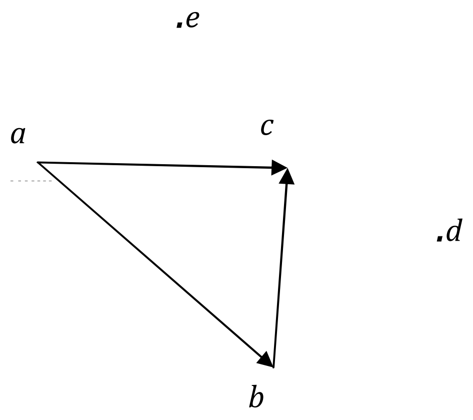


Fig. 2.4. Graph relation R (*Example 2.13*)

Solution

There is no the best elements since the element e is not comparable with the others. The worst elements also do not exist. The elements a, d and e are maximal with the respect to the relation R^S. Minimal elements are c, d, e.

Note that the elements d, e are maximal and minimal at the same time. This is due to the fact that they are not comparable to others, i.e. we do not have information on the benefits of these elements.

Definition 2.30. The set $\max_R X$ of maximal objects of the set X in relation to R is internally stable in sense that when $a, b \in \max_R X$, then none of the statements can be fulfilled: $a\,R\,b$ and $b\,R\,a$.

Definition 2.31. A set is called an *externally stable* if for each non-maximal element $a \in X$ there is a more preferred element of it among the maximal, i.e. there will be a fair statement: $a^0 R\, a$ for some element $a^0 \in \max_R X$.

An internally and externally stable set $\max_R X$ is called the *kernel* of the relation R in the set X.

The notion of stability has great importance because if the set $\max_R X$ is externally stable, then the optimal element must be chosen from this set. If it is not externally stable, then there is no reason to restrict its choice.

When there is a need to choose more than one best elements or to arrange all objects against their advantages, then the notions of the maximal element and the kernel of the relation lose their meaning.

Example 2.14. Assume that the set $B = \{a, b, c\}$, and it has the relation: $R = \{(a, c)\}$. Here is the set of maximal elements $\max_R B = \{a, b\}$. However, when choosing two of the best elements, one should not take into account the presence of element c, because if information appears it is more preferable than b, then elements a and c will be desired.

Definition 2.32. The numeric function φ defined in the set X is called *increasing* (non-decreasing) by the relation R, when the condition $a\,R\,b$ implies that $\varphi(a) > \varphi(b)$ [respectively $\varphi(a) \geq \varphi(b)$] for all elements $a, b \in X$.

There is the following statement:

Lemma 2.1. Let the set $B \subseteq A$ and element $a^0 \in B$ give a non-decreasing by relation *R* in the set *B* of the function Ψ of the largest value in it. Then, in order for an object a^0 to be maximal in relation to the *R* relation in the set of *B*, one of the following conditions is sufficient:

1. Ψ increases by the relation *R* in the set *B*.
2. $a^0 \in B$ is the unique point of the maximum of function Ψ in the set *B*.

Proof

Assume that the element a^0 is not maximal in relation *R*, then, in set *B* there is an element *a*, which prevails a^0 over the relation *R* that is $a\,R\,a^0$. But in this case, there must be a strict inequality: $\Psi(a) > \Psi(a^0)$ since function Ψ increases by the relation *R* in the set *B*. But the strict inequality contradicts the fact that the element a^0 is the point of the maximum of function Ψ and the non-uniform inequality: $\Psi(a) \geq \Psi(a^0)$ because a^0 represents a single point of maximum Ψ in the set *B*. The proof is complete.

When simulating real systems, there may be the situations where DM or experts have no clear picture of the benefits between alternatives, but they definitely need to make specific conclusions about which of the alternatives is the best. In this case, experts are forced to somehow "coagulate" their knowledge and representation, and the corresponding mathematical model will be less adequate to the actual situation.

A more flexible way of formalizing such representations is the ability for experts to determine their degree of belief in the superiority of the alternative using numbers from the interval [0;1], that is to describe their arguments with the help of a fuzzy preference relation, when each pair of alternatives (x, y) corresponds to the number from the interval [0,1], which reflects the measure of the correctness of the benefit: $x \geq y$. Methods of decision-making based on fuzzy preference will be discussed later in Section 5.

Note that the characteristic feature of the "language" of binary relations is the assumption that the result of the comparison of the merits of the two elements does not depend on the composition of the whole set. However, in some cases, such an addiction takes place, and for its inclusion a more rich "language" of the description of the benefits is required, based on the use of the choice functions.

2.7 Concept of choice function. Classes of choice functions

In real situations of choice in the set of alternatives Ω, a decision-maker chooses an alternative guided by their personal opinion about the best alternatives.

Different people have different ideas about the same situation. However, it is logical to assume that under similar conditions one and the same person will act in the same way. Therefore, it is possible to formulate a rule with the help of which a choice will be made.

Let us consider the following situation: let Ω be the set of alternatives among which the choice is made, and the set of alternatives *X* represent its subset.

Let's denote the set of alternatives that DM allocates from the set X by $C(X)$.

For example, Ω is the set of all groups in a higher educational institution, and *X* is an arbitrary subset Ω, which may be a set of third-year groups, a set of faculty groups etc. Let's consider that $C(X)$ is the best group in the set of groups *X*. Regardless of who is a decision maker (a person who chooses the best group), it is naturally to consider that the best group in the institution will be the best group of its year of study, its faculty etc.

Mathematically this can be written as following: if $X' \subset X$ and $x \in C(X) \cap X'$, then $x \in C(X')$.

Consequently, all sorts of choices in a particular situation can be considered logically grounded, if in other situations associated with this one, solutions are known. This means that the sets $C(X)$ are dependent on different sets *X* *i*f the choice is made by the same DM. For formalization of this dependence, the concept of the choice function is used.

Definition 2.33. The *choice function* $C(X)$ is called the mapping which sets its subset $C(X) \subset X$ to each of sets $X \subset \Omega$.

The set $C(X)$ will be interpreted as the most preferred alternatives from the set *X*.

It is necessary to note that in this definition there are no prior restrictions of a choice function, in particular, the possibility of an empty choice that is, a situation when $C(X) = \varnothing$ is not excluded. This situation is called a *rejection of choice*. An example of the rejection of choice can be the case, when a buyer leaves a shop, having bought nothing.

In a particular case, especially, when the known relation *R* of the strict preference in the set of alternatives is known, the choice function can be defined by the following equality:

$$C(X) = \max_R X.$$

Example 2.15. Let the preference relation *R* be given in the set: $X = \{x_1, x_2, x_3, x_4\}$ by the matrix, namely:

$$R=\begin{pmatrix}1&1&1&1\\0&1&0&0\\1&1&1&0\\1&0&1&1\end{pmatrix}.$$

To construct a selection function corresponding to this relation.

Solution

We construct a relation of strict preference: $R^S = R \setminus R^{-1}$, which corresponds to this relation:

$$R^s=\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}.$$

Now we will set the choice function, applying the following rule: $C(X)=\max_R X$. For this purpose we consider all possible subsets of the given set $\{x_1,x_2,x_3,x_4\}$ and define the maximal elements by narrowing the relation R to the corresponding subset.

Firstly, let's consider one-element subsets. The choice from one element will be the element exactly. Therefore,

$$C(\{x_1\})=\max_R\{x_1\}=x_1,$$

$$C(\{x_2\})=\max_R\{x_2\}=x_2,$$

$$C(\{x_3\})=\max_R\{x_3\}=x_3,$$

$$C(\{x_4\})=\max_R\{x_4\}=x_4$$

Next, we consider the two-element subset. The narrowing of this relation to the set $\{x_1,x_2\}$ gives an opportunity to conclude that element x_1 is more prevalent than x_2, therefore, the maximum element for this set will be x_1, and then

$$C(\{x_1,x_2\})=\max_R\{x_1,x_2\}=x_1.$$

Similar to other two-element sets:

$$C(\{x_1,x_3\})=\max_R\{x_1,x_3\}=\{x_1,x_3\},\quad C(\{x_1,x_4\})=\max_R\{x_1,x_4\}=\{x_1,x_4\},$$

$C(\{x_2,x_3\}) = \max_R\{x_2,x_3\} = \{x_3\}$, $\quad C(\{x_2,x_4\}) = \max_R\{x_2,x_4\} = \{x_2,x_4\}$,

$C(\{x_3,x_4\}) = \max_R\{x_3,x_4\} = \{x_4\}$

and the same to:

$C(\{x_1,x_2,x_3\}) = \max_R\{x_1,x_2,x_3\} = \{x_1,x_3\}$,

$C(\{x_1,x_2,x_4\}) = \max_R\{x_1,x_2,x_4\} = \{x_1,x_4\}$,

$C(\{x_1,x_3,x_4\}) = \max_R\{x_1,x_3,x_4\} = \{x_1,x_4\}$,

$C(\{x_2,x_3,x_4\}) = \max_R\{x_2,x_3,x_4\} = \{x_4\}$,

$C(X) = C(\{x_1,x_2,x_3,x_4\}) = \max_R\{x_1,x_2,x_3,x_4\} = \{x_1,x_4\}$.

So, the choice function has been given.

Note that there are other ways to specify the choice function.

Thus, in terms of the relation of preference, we can construct the choice function, but there are no corresponding preference relation for any choice function,

Example 2.16. The choice function is given as follows:

$C(\{x_1\}) = x_1$, $\quad C(\{x_2\}) = x_2$, $\quad C(\{x_3\}) = x_3$,

$C(\{x_1,x_2\}) = x_1$, $\quad C(\{x_1,x_3\}) = \{x_1,x_3\}$,

$C(\{x_2,x_3\}) = \{x_2\}$, $\quad C(\{x_1,x_2,x_3\}) = \{x_1,x_3\}$.

As we see, the last two conditions contradict each other. So, it is impossible to construct a relation.

Example 2.17. The choice function is given as follows:

$C(\{x_1\}) = x_1$, $\quad C(\{x_2\}) = x_2$, $\quad C(\{x_3\}) = x_3$,

$C(\{x_1,x_2\}) = x_1$, $\quad C(\{x_1,x_3\}) == \{x_1,x_3\}$, $\quad C(\{x_2,x_3\}) = \{x_3\}$,

$C(\{x_1,x_2,x_3\}) = \{x_1,x_3\}$.

The following strong-preference relation will correspond to this function:

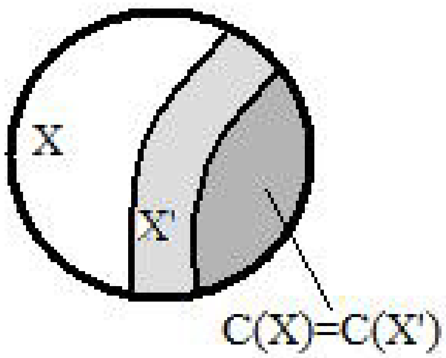
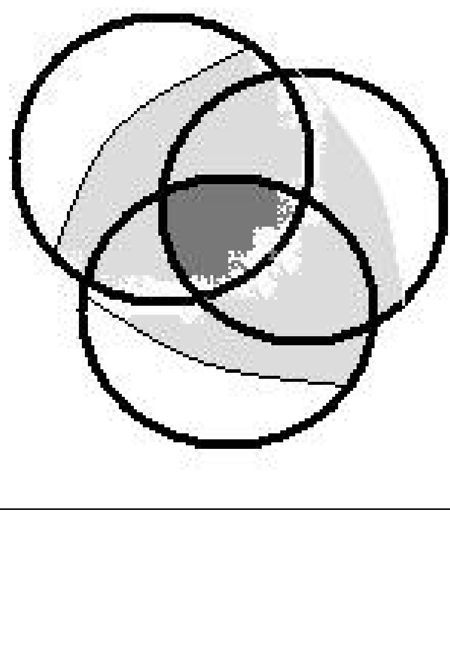
$$R^s = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The choice functions are conveniently to categorize according to the conditions that are commonly used during their study.

Examples of such conditions are given in Table 2.2.

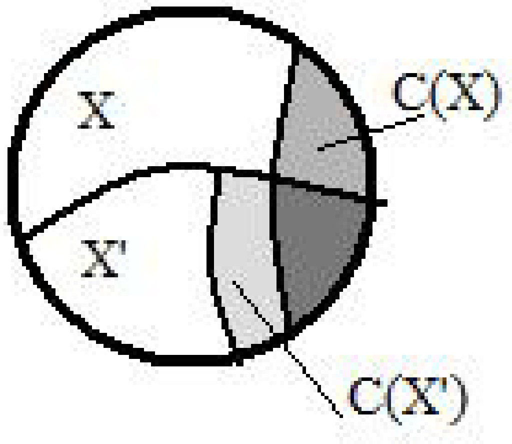
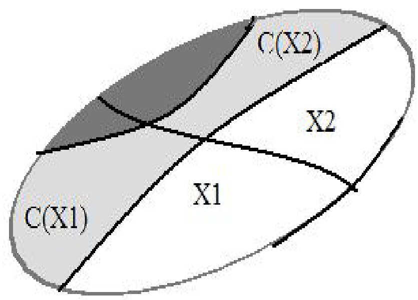
Table 2.2.

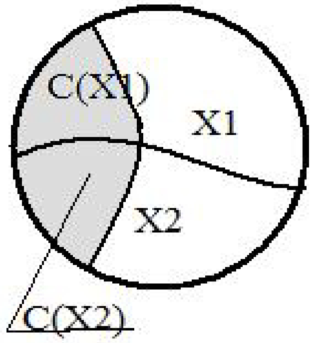
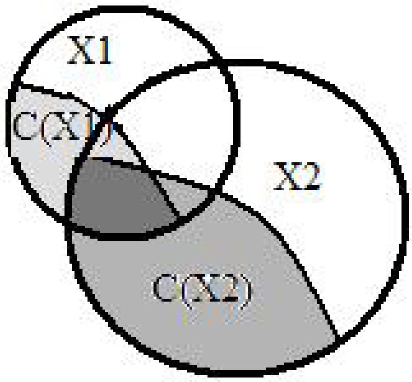
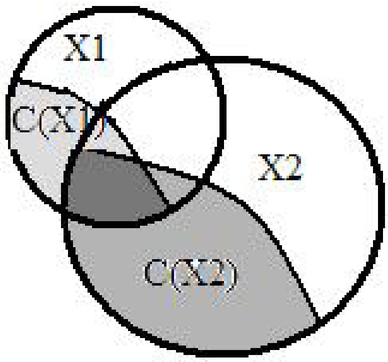
Classification of choice functions

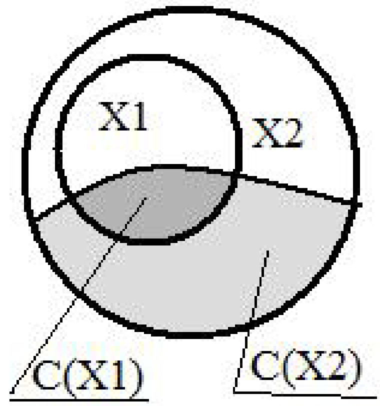
1. Condition of independence from neglected alternatives	
If $C(X) \subset X' \subset X$, then $C(X) = C(X')$. The condition is interpreted as follows: when considering an arbitrary set X' containing all the alternatives selected from the set *X*, then the choice X' is the same as the choice from the set *X*. For example, when during the competition a project *x* was not included in the best ones, then the list of winners will not change in the other competition, where all the projects which participated in the previous one with the exception of *x* take part..	
2. Condition of agreement	
$\bigcap_i C(X_i) \subset C\left(\bigcup_i X_i\right)$ This condition means that the alternatives which have been selected from each set X_i will also be selected from their association.	

Continuation the Table 2.2.

<table>
<tr><th colspan="2">3. The condition of imitation</th></tr>
<tr><td>If $X' \subset X$, then
$C(X') \supset C(X) \cap X'$

The meaning of this condition is following: if we consider the choice from an arbitrary set and choose from a subset of it, then all the alternatives which have been selected from the initial set and are included in the subset being considered will also be selected from this subset.
For example, if the project from Bulgaria was the winner, at an international competition, then it should be among the winners of a Bulgarian competition of projects.</td><td>
</td></tr>
<tr><th colspan="2">4. Plott’s condition (independence from path selection)</th></tr>
<tr><td>$C(X_1 \cup X_2) = C(C(X_1) \cup C(X_2))$
Plott's condition implies that the choice of alternatives from the union of choices, which in their turn are made from each set, exactly corresponds to the choice from choice associations, which are made from each set separately
For example, to conduct an international competition, you can first select the winners of the national competitions, and then arrange competitions among them.</td><td>
</td></tr>
</table>

5. The condition of futility	
$C(X_1 \cup X_2) = C(X_1) \cup C(X_2)$ This condition means that the choice of aggregation of sets equals the combination of choices from each set separately. For example, people from the different organizations have been honoured on the district board of honour	
6. The multiplicity condition	
$C(X_1 \cap X_2) = C(X_1) \cap C(X_2)$ Under this condition, the selection from the intersection of the two sets will be equal to the intersection of the choices.	
6. Multiplicity condition	
$C(X_1 \cap X_2) = C(X_1) \cap C(X_2)$ Under this condition, the choice from the intersection of the two sets will be equal to the intersection of the choices.	

7. Conditions of monotony	
If $X_1 \subset X_2$, then $C(X_1) \subset C(X_2)$, i.e. a choice from a wider set will be wider.	

2.8 Utility functions

To compare different alternatives and choose the best of them, you can also use some quantitative measure of their properties, with the meanings of which one can compare alternatives with each other and choose the best one. The decision rules (procedures) based on this measure use the utility hypothesis developed by J. von Neumann and O. Morgenstern [26]. The mathematical basis of this hypothesis is the axioms system, which asserts the existence of some degree of value that allows to streamline alternatives (results of decisions). This measure is called the *utility function*, or the *usefulness* of the results.

The practical application of the utility theory is rooted in the following axioms:

1. The result (alternative) x_i is better than the alternative x_j (written as $x_i > x_j$ only if $u(x_i) = f(x_i) > u(x_j)$, where $u(x_i)$ i $u(x_j)$ is the utility value of the alternatives x_i and x_j, respectively.

2. If $x_i > x_j$ and $x_j > x_k$, then $x_i > x_k$ and $u(x_i) > u(x_k)$ (transitivity).

3. If x_1, x_2 are some alternatives, then $u(x_1, x_2) = u(x_1) + u(x_2)$ (additivity).

Similarly, when there are n results x_1, x_2, … x_n, which are achieved simultaneously, then

$$U(x_1, x_2, \dots x_n) = \sum_{i=1}^{n} u_s(x).$$

In other words, the usefulness of several results that are achieved simultaneously equals to the sum of their utility values.

With application of the notions of utility function (objective function) $f(x)$ we define the following relations in the set of alternatives X:

– the relation of the weak (non-strict) preference is "not worse", which is indicated by the symbol « $\geq$ »;

– the equivalence relation denoted by the « ~ » symbol;

– the relation of the strong preference, which is indicated by the symbol « > ».

Definition 2.34. For two alternatives x_1, x_2 it can be stated that

$$x_1 \geq x_2, \text{ only if } f(x_1) \geq f(x_2);$$

$$x_1 \sim x_2, \text{ only if } f(x_1) = f(x_2);$$

$$x_1 > x_2, \text{ only if } f(x_1) > f(x_2).$$

The symbols « $\geq$ » and « > » when comparing the values of the functions for different alternatives are taken depending on whether the best alternative is considered to be with a greater or lesser value of the objective function.

The method of determining usefulness of possible results is developed in the Manual [1].

Let's consider several variants of application of this method in different situations.

I. Only two results are available.

In this case, the utility calculation methodology is:

1. Determine which result is the best for the decision-maker. Assume that $x_1 > x_2$, that is, the alternative x_1 is better than the alternative x_2.

2. Then, we define the probability α in which the achievement of the result x_1 will be equivalent to the result x_2, obtained with probability 1.

3. Evaluate the relation between the utility values of the results x_1 and x_2.

For this we assume that the utility $u(x_2) = 1$,

then $\alpha u(x_1) = u(x_2)$; $u(x_1) = 1/\alpha$.

II. There are n possible alternatives x_1, x_2, … x_n, among which there are preferences: $x_1 > x_2 > \ldots > x_n$.

In this case, the method of determining the usefulness is:

1. Determine the value of α_1 from the condition that $\alpha_1 u(x_1) = u(x_2)$.

2. Similarly calculate that

$$\alpha_2 u(x_2) = u(x_3);$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\alpha_{n-1} u(x_{n-1}) = u(x_n).$$

3. Assuming that the usefulness of the least significant result is 1, we find the value of utility for other results, namely:

$$u(x_n) = 1;$$

$$u(x_{n-1}) = 1/\alpha_{n-1};$$

$$\cdots\cdots\cdots\cdots$$

$$u(x_1) = \frac{1}{\alpha_{n-1}\alpha_{n-2}\ldots\alpha_1}.$$

III. Available qualitative criteria. Under these conditions, we have information about the benefits of individual alternatives and their groups. Then, a methodology based on the algorithm proposed by R. Acoff and R. Churchman [1] can be applied.

Assume that there are n alternatives: x_1, x_2, … x_n. The method of determining the utility involves the following steps:

1. Arrange all alternatives for diminishing utility. Let x_1 be the alternative that has the greatest preference, and x_n is an alternative, the preference of which is the smallest.

2. Make a table of possible combinations of the results that are achieved simultaneously, and then set their preference over individual results x_1, x_2, … x_n (Table 2.3).

Information about the benefits of results is usually obtained from experts.

Table 2.3

1	x_1 or $x_2 + x_3 + \ldots + x_n$	$n + 1$	x_2 or $x_3 + x_4 + \ldots + x_{n-1}$
2	x_1 or $x_2 + x_3 + \ldots + x_{n-1}$	$n + 2$	x_2 or $x_3 + x_4 + \ldots + x_{n-2}$
3	x_1 or $x_2 + x_3 + \ldots + x_{n-2}$	$n + 3$	x_2 or $x_3 + x_4 + \ldots + x_{n-3}$
…	…	…	…
n	x_2 or $x_3 + x_4 + \ldots + x_n$	N	x_{n-2} or $x_{n-1} + x_n$

3. Assign individual results $u_0(x_1)$, $u_0(x_2)$, ... , $u_0(x_n)$ to the initial estimates of the utility. Then the initial estimates are subjected to the last correlation Table. 2.3. If it is satisfied, the estimates do not change.

In the opposite case, carry out a correction of utility so that this relation is satisfied.

After that, proceed to the next relation. The correction process continues until a system of estimates $u^*(x_1)$, $u^*(x_2)$, … $u^*(x_n)$ is obtained, that satisfies all the relations listed in the table. The correction should be made in such a way that it would be necessary to change the minimum number of evaluations of the results.

Example 2.18. Let an expert to arrange five results $x_1, x_2, \dots x_5$, assigning them the following estimates: $u_0(x_1) = 7$; $u_0(x_2) = 4$; $u_0(x_3) = 2$; $u_0(x_4) = 1{,}5$; $u_0(x_5) = 1$.

Having considered the possible choices, s/he expressed the following thoughts on the value of a combination of options:

1) $x_1 < x_2 + x_3 + x_4 + x_5$;
2) $x_1 < x_2 + x_3 + x_4$;
3) $x_1 > x_2 + x_3 + x_5$;
4) $x_1 > x_2 + x_3$;
5) $x_2 < x_3 + x_4 + x_5$;
6) $x_2 > x_3 + x_4$;
7) $x_3 > x_4 + x_5$.

It is necessary to evaluate usefulness of the results so as to satisfy all the inequalities.

To solve this problem, we substitute the initial estimates for inequality 7 that is

$$u_0(x_3) = 2 < u_0(x_4) + u_0(x_5) = 2{,}5.$$

Consequently, inequality 7 is not satisfied.

Let's change the utility of the result x_3: $u_1(x_3) = 3$ and check the inequality 6. So,

$$u_0(x_2) = 4 < u_1(x_3) + u_0(x_4) = 4{,}5.$$

This inequality is also not satisfied.

We assume that $u_1(x_2) = 5$, then inequality 5 is satisfied.

We turn to inequality 4:

$$u_0(x_1) = 7 < u_1(x_2) + u_1(x_3) = 8.$$

It doesn't work, i.e. is not executed, so let us assume that $u_1(x_1) = 8{,}5$. Now the inequalities 3, 2, 1 are satisfied.

Check again the inequalities 6 and 7 with changed values of utility alternatives:

$$5 > 3 + 1{,}5,$$

$$3 > 1{,}5 + 1.$$

Thus, both inequalities are fulfilled.

So, let's write down the final estimates of the usefulness of the results:

$$u_1(x_1) = 8{,}5;\ u_1(x_2) = 5;\ u_1(x_3) = 3;\ u_1(x_4) = 1{,}5;\ u_1(x_5) = 1.$$

Note that the described utility method can be used when the number of results is limited, namely $n < 6$ or 7.

In cases where $n > 7$, a modified method of correction of estimates [1] is proposed.

The set of alternatives is divided into subsets, which consist of 5 – 7 alternatives and have one common result, for example x_1. Then the initial values of utility are assigned to all the alternatives, and the usefulness of the common result x_1 must be the same in all subsets. Next, the method of correction of utility evaluations is applied separately to each of the subsets, taking into account the constraints: $u(x_1) = \text{const}$. As a result, a system of utility with one measure for all subsets $u(x_1)$ is obtained.

Once, the utility function of all alternatives is defined in accordance with the described methodology, the deciding rule for choosing the best of them in terms of certainty is written as follows: find an alternative x_0 so that $f(x_0) = \max f(x)$.

Obviously, the objective function, on the basis of which the desired alternative is chosen, can be constructed in different ways.

Definition 2.35. Objective functions $f_1(x)$ and $f_2(x)$ that characterize the same feature of a being chosen solution, which are defined in one set of valid alternatives, will be called *equivalent* if they specify the same relation of weak preference that is, when for any two alternatives x_1 and x_2 from the statement: $x_1 \overset{f_1}{\geq} x_2$ brings to that $x_1 \overset{f_2}{\geq} x_2$, and on the contrary, when the statement: $x_1 \overset{f_2}{\geq} x_2$ turns out that $x_1 \overset{f_1}{\geq} x_2$. Here the index f_i over the sign of a weak preference indicates the function by which this relation is given.

From this definition it turns out that the equivalent objective functions set in set X are the same relations of strict preference and equivalence. The simple theorem shown below determines which features/properties must satisfy the equivalent objective functions [22].

Theorem 2.1. To ensure that the objective functions $f_1(x)$ and $f_2(x)$ are equivalent, there is a sufficient existence of a monotonic transformation $w(z)$ capable of transferring the value domain of the function $f_2(x)$ to the domain of the values of the function $f_1(x)$, ie $f_1(x) = w(f_2(x))$ for the whole set of admissible alternatives. In this case, if both objective functions are maximized, then the transformation $w(z)$ must be a monotonically increasing function, and if not, $w(z)$ must be a monotonically decreasing function.

Proof

Consider the case when the criteria are maximized and the transformation $w(z)$ is monotonously increasing since the other cases are proved in the same way. Then, if $x_1 \overset{f_2}{\geq} x_2$, i.e. $f_2(x_1) \geq f_2(x_2)$, then $w(f_2(x_1)) \geq w(f_2(x_2))$. So, $x_1 \overset{f_1}{\geq} x_2$.

The statement: $x_1 \overset{f_2}{\geq} x_2$ follows that $x_1 \overset{f_1}{\geq} x_2$ of the monotony of the converse transformation.

The theorem has been proved.

Here are examples of equivalent maximized objective functions:

$$f_1(x) = af_2(x) + b, \quad \text{where } a > 0,$$

$$f_1(x) = ln\, f_2(x) + b, \ \text{ if } f_2(x) > 0.$$

Conclusions

The concept of a binary relation allows formalizing operations of pairwise comparison of objects and mathematically substantiating the choice of one or more objects in the event that it is impossible to set a criterion on the set of alternatives, but one can evaluate the preferences of one alternative over another.

Binary relations can be define using matrix, graph, or cuts. They are used for crossing, union, complementary and others.

In the decision-making theory, the following features of relations are important as reflexivity, symmetry (asymmetry), transitivity.

The choice functions are used to specify the rules for choosing alternatives. Depending on the nature of the task, these functions may have different properties and features. Using this relation of preferences, one can construct the corresponding choice function, but not vice versa.

Utility functions represent a quantitative measure by which alternatives can be compared.

SELF-STUDY

Questions for assessment and self-assessment

1. Give a definition of a binary relation.
2. What are the ways of setting up a relation?
3. How to set up a matrix relation?
4. How to set a graph as a relation?

5. How the relations are determined by cuts?
6. Formulate the definition of the upper (lower) cuts of the relation.
7. Which methods of assigning relations can be used in an infinite set of elements?
8. What mathematical operations are performed on relations?
9. What relation is called *reflexive (irreflexive)*?
10. What relation is called *symmetric, antisymmetric, asymmetric*?
11. What relations are called *transitive, highly transitive, negative transitive*?
12. How to calculate the transitive closure relation?
13. What properties are the characteristics of the relation of preference?
14. Give the definition of the best (worst) element of the set.
15. What element of the set is called the minimal (maximal) for this relation of preference?
16. What is the meaning of the decision-making theory in terms of the best, the worst, the maximal and the minimal elements? Where are they used?
17. Give the definition of equivalence, indifference, dominance.
18. How the given relation of the non-substantial preference can be used to construct the corresponding relations of strict preference, indifference, equivalence?
19. What do the features of the external and internal stability of the set mean?
20. Give the definition of the choice function.
21. How can we construct the choice function with given preference relation?
22. Is it always possible to construct an appropriate preference relation according to the choice function?
23. Which features and properties contribute to choice function classification?
24. Give examples of the conditions under which the choice functions are classified.
25. What does a condition of imitation mean? Summary? Plott?
26. What is a utility function?
27. How is the utility of alternatives determined against the given benefits?
28. Formulate an algorithm for constructing a utility function in the set of alternatives when qualitative criteria are present.
29. What objective functions are called the equivalent ones?
30. What properties or features should satisfy the equivalent objective functions?
31. Give examples of equivalent transformations of the objective functions.

Hands – on practice

Task A.

1. The relation is given in the form of a matrix. Define it with: a) a graph; b) the upper cuts; c) the lower cuts.

$$R_1 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

2. The relation is given as a matrix. Define it with: a) a graph; b) the upper cuts; c) the lower cuts.

$$R_2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

3. Define the relation "less than or equal" in the set of integers from one to ten using a matrix.

4. In the set: $X = \{a, b, c, d\}$, the relation R is given by the graph (Figure 2.5). Define it with: a) a matrix; b) the upper cuts; c) lower cuts.

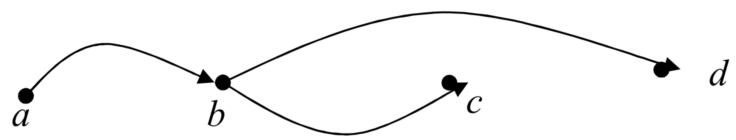


Fig. 2.5. Graph of the relation *R*

5. In the set: $X = \{a, b, c, d\}$, the relation R is given by the graph (Figure 2.6). Define it with: a) a matrix; b) the upper cuts; c) lower cuts.

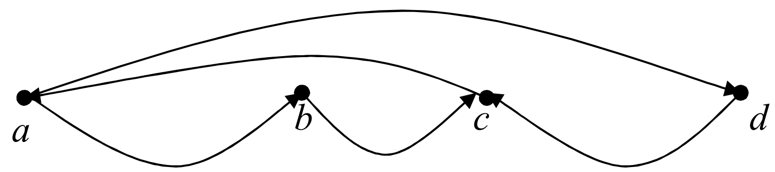


Fig. 2.6. Graph of the relation *R*

6. Check the properties of the relations listed below.

$$a)\ R=\begin{pmatrix}1&0&0&1\\0&1&1&1\\0&1&1&1\\0&1&0&1\end{pmatrix};\qquad b)\ R=\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{pmatrix};$$

$$c)\ R=\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&1\\0&1&0&0\end{pmatrix};\qquad d)\ R=\begin{pmatrix}1&0&1&1\\0&0&1&1\\1&1&0&0\\1&1&0&1\end{pmatrix}.$$

7. Using the conditions of task 6, *a - d*, determine complementary and converse relations.

8. Determine the intersection and union of the following relations.

$$R_1=\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{pmatrix};\qquad R_2=\begin{pmatrix}1&0&1&1\\0&0&1&1\\1&1&0&0\\1&1&0&1\end{pmatrix}.$$

9. Using the conditions of task 6, *a – d*, construct a relation of strict preference, equivalence, indifference.

10. Find the largest, smallest, maximal and minimal elements by relation from task 6, a - d (if any exist).

11. Constuct a choice function using given relation of preference.

$$a)\ R=\begin{pmatrix}1&0&0&1\\0&1&1&1\\0&1&1&1\\0&1&0&1\end{pmatrix};\qquad б)\ R=\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{pmatrix}.$$

12. Evaluate the utility of the results according to the preferences, if $x_1 > x_2 > \ldots > x_5$ and give the preferences of the results: $\alpha_1 = 2$, $\alpha_2 = 3$, $\alpha_3 = 2$, $\alpha_4 = 1{,}5$.

13. Construct a preference relation corresponding to the option selected below (if possible).

$a)$ $C(\{a\}) = \{a\}, C(b) = \{b\}, C(\{c\}) = \{c\}, C(\{a,b\}) = \{b\}, C(\{a,c\}) = \{a\},$
$C(\{b,c\}) = \{b\}, C(\{a,b,c\}) = \{c,b\};$

$b)$ $C(\{a\}) = \{a\}, C(\{b\}) = \{b\}, C(\{c\}) = \{c\}, C(\{a,b\}) = \{a\}, C(\{a,c\}) = \{a\},$
$C(\{b,c\}) = \{b\}, C(\{a,b,c\}) = \{a\}.$

14. Let an expert arrange five results $x_1, x_2, \ldots x_5$, assigning them the following estimates: $u_0(x_1) = 10$; $u_0(x_2) = 5$; $u_0(x_3) = 3$; $u_0(x_4) = 2$; $u_0(x_5) = 1$.

Having considered the possible choices, s/he expressed the following thoughts on the value of a combination of options:

$x_1 \le x_2 + x_3 + x_4 + x_5;$

$x_1 \ge x_2 + x_3 + x_4;$

$x_1 \ge x_2 + x_3 + x_5;$

$x_1 \ge x_2 + x_3;$

$x_2 \ge x_3 + x_4 + x_5;$

$x_2 \ge x_3 + x_4;$

$x_3 \le x_4 + x_5.$

Evaluate the benefits of the results.

15. Let an expert arrange five results $x_1, x_2, \ldots x_5$, assigning them the following estimates: $u_0(x_1) = 8$; $u_0(x_2) = 6$; $u_0(x_3) = 2$; $u_0(x_4) = 1{,}5$; $u_0(x_5) = 1$.

Having considered the possible choices, s/he expressed the following thoughts on the value of a combination of options:

$x_1 \le x_2 + x_3 + x_4 + x_5;$
$x_1 \le x_2 + x_3 + x_4;$
$x_1 \le x_2 + x_3 + x_5;$
$x_1 \ge x_2 + x_3;$
$x_2 \ge x_3 + x_4 + x_5;$
$x_2 \ge x_3 + x_4;$
$x_3 \le x_4 + x_5.$

Evaluate the usefulness of the selection results.

Task B

1. Describe with the help of a matrix the relation "object x consumes the object y " in the set: A = {man, tiger, gooseberry, pike, ram, gazelle, wheat, wild boar, clover, field mice, snake, acorn, crucifix}. What properties are specific for this relation, i.e. its feature(s) or property? Could it be called a superiority? Equivalence? Relation to the order?

2. With the help of the matrix or the graph the relation specifies "operation x should be performed after the operation y" in a set of repairs. What properties characterize this relation? Could it be called a preference? Equivalence? Relation to the order?

SECTION 3

MULTI-CRITERIA OPTIMIZATION PROBLEM

In this section you will:

- study the features of decision-making problems in the presence of many criteria;
- learn and practice methods of multi-criteria optimization.

3.1 General statement of multi-criteria optimization problem

As it was mentioned above, one of the problems in decision-making is the presence of many criteria which are not always consistent with each other that leads to the construction of appropriate mathematical models and the usage of certain mathematical methods. One way of formalizing such problems is the usage of multi-criteria optimization models.

In this section, we discuss finite-dimensional multi-criteria tasks, i.e. those in which the set of alternatives X belongs to finite-dimensional space E_m, and set vector criterion

$$f(x) = (f_1(x), \dots f_M(x)).$$

The set X usually stands out from wider set D with the help of special constraints, which is often served in the form of inequalities, namely:

$$X = \{x \in D | g_1(x) \geq 0, \dots, g_k(x) \geq 0\},$$

where g_i ($i = 1, 2,\dots k$) is numerical functions defined in the set D, forming a vector-constraint function.

Depending on the structure of the set X (or D) and the properties of the objective functions $f_j(x)$ (and g_i)) for the convenience of the study, different classes of multi-criteria tasks are distinguished. If the set X is finite, then the problem is called finite, but when this set is finite or read, then the problem is referred to as discrete, if all the components of x_i alternatives $x \in X$ are integers – the problem is called integer. Boolean, linear, concave and other multi-objective optimization problems are defined accordingly.

For example, consider this problem:

Imagine you are given a set of alternatives X which properties are described by the set of objective functions: $f = \{f_i(x)\}$, $i \in I$, $x \in X$, where I is the set of indices $I = \{1, 2, \dots, M\}$. We assume that m of the first objective functions are maximized and the other $(M - m)$ ones are minimized. Let us denote by I_1 the set of indices for which the objective functions are maximized, so $I_1 = \{1, 2, \dots, m\}$; I_2 is the set of indices for which the objective function is minimized: $I_2 = \{m+1, m+2, \dots M\}$. Then, a multi-objective problem can be written as follows:

$$\begin{aligned} &f_i(x) \to \max,\ i \in I_1, \\ &f_i(x) \to \min,\ i \in I_2, \\ &\quad x \in X. \end{aligned} \tag{3.1}$$

3.2 The concept of an effective alternative

Consider the problem of multi-criteria optimization (3.1). Given the objective functions, alternatives x_1 and x_2 can be compared in this way:

– alternative x_1 is not worse than alternative x_2 ($x_1 \geq x_2$), when

$$\begin{cases} f_i(x_1) \geq f_i(x_2),\ i \in I_1, \\ f_i(x_1) \leq f_i(x_2),\ i \in I_2; \end{cases}$$

– alternative x_1 is equivalent to alternative x_2 ($x_1 \sim x_2$), if

$$f_i(x_1) = f_i(x_2), \forall i \in I;$$

– alternative x_1 is strictly dominated by an alternative x_2 ($x_1 > x_2$), when

$$\begin{cases} f_i(x_1) \geq f_i(x_2), i \in I_1, \\ f_i(x_1) \leq f_i(x_2), i \in I_2, \end{cases}$$

and at least one inequality is performed as strict.

Obviously, not every pair of alternatives can be compared with each other.

Example 3.1. Let $f(x) = (f_1(x), f_2(x))$, where the function $f_1(x)$ is maximized and the function $f_2(x)$ is minimized on the discrete set $X = \{x_1, x_2, x_3, x_4, x_5\}$. The values of the objective functions on the set X are presented in Table 3.1.

Table 3.1

Function value: $f(x) = (f_1(x), f_2(x))$

	$f_1(x)$	$f_2(x)$
x_1	7	5
x_2	6	2
x_3	5	4
x_4	6	6
x_5	4	1

Definition 3.1. The alternative x_0 is called *efficient* if there is no alternative X in the set of admissible alternatives x, that satisfies the following inequalities:

$$f_i(x) \geq f_i(x_0), \quad i \in I_1,$$
$$f_i(x) \leq f_i(x_0), \quad i \in I_2,$$

and at least one of them is performed as a strict.

In other words, no other alternative can "improve" any value of the objective function without worsening the value of some other. That is why sometimes an effective alternative is called *non-improved* for the set of purposes or Pareto *optimal*.

So, there is a need to search the solution of a problem of multicriteria optimization among the set of Pareto optimal alternatives. However, we cannot say which alternative to choose as the further research is needed.

It follows from the definition of effective alternatives that they may not be comparable. In this regard, the following statement is true:

Lemma 3.1. Two effective alternatives are either equivalent or incomparable.

Proof

If x_0 is an effective alternative, then for any alternative x', comparable to x_0 in the set of objective functions, or n equalities are fair: $f_i(x') = f_i(x_0), \forall i \in I$, and then the alternative x' is equivalent to x_0 or an index $s \in I$ can be found, for which $f_S(x_0) \geq f_S(x')$ when $s \in I_1$, or $f_S(x_0) \leq f_S(x')$ if $s \in I_2$, then the alternative x', cannot be effective.

The Lemma is proved.

From this Lemma it turns out that when there is only one effective alternative, it gives an optimum to each of the criteria.

If there are two or three criteria, the set of effective alternatives can be graphed.

For example, consider a problem that includes two criteria, each of which is maximized, and the set of valid alternatives in the criterion space has the form shown in Figure 3.1. The Pareto set in this case (if $m = 2$) is figuratively spoken as the North-Eastern boundary of the set of feasible solutions without those parts of it that are parallel to the coordinate axes or lie in sufficiently deep and steep dips (this set is shown by a thick line in the Figure 3.1).

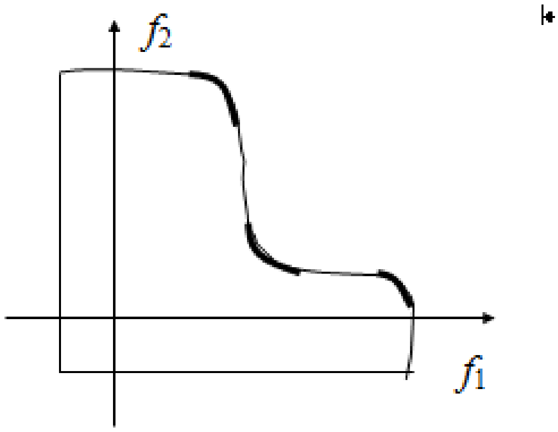


Fig. 3.1. Graphical representation of the set of valid solutions and Pareto set

Definition 3.2. The alternative (solution) is referred to as *weakly effective,* and also *weakly optimal in Pareto*, or *optimal in Slater* when there is no other alternative (solution) for which:

$$f_i(x) > f_i(x_0), \quad i \in I_1,$$
$$f_i(x) < f_i(x_0), \quad i \in I_2.$$

Weakly effective alternative is an estimate of the maximum with respect to relation " > " in contrast to effective alternatives that are maximal with respect to relation " ≥ ".

Note that any effective alternative is weakly effective and accordingly the set of effective alternatives $P(Y)$ is contained in the set of weakly effective alternatives $S(Y)$.

Many effective alternatives $P(Y)$ [weakly effective alternatives $S(Y)$] is called *externally stable* if for any element of $y \in Y \backslash (P(Y)$ [$y \in Y \backslash S(Y)$], such estimate as $y^0 \in P(Y)$ [accordingly $y^0 \in S(Y)$], for which $y^0 \geq y$ ($y^0 > y$) exists.

If the set Y consists of a finite number of estimates, then the sets $P(Y)$ and $S(Y)$ will be externally stable. When the set Y is infinite, the sets of effective alternatives $P(Y)$ and $S(Y)$ may not be externally stable. However, these sets will be externally stable under usual assumptions for optimization problems (X – compact, the function f – semi-continuous from above).

Example 3.2. Let Y be the unit square from which the upper right vertex has been "removed " (Figure 3.2).

For such a set Y, the set $P(Y)$ is obviously empty, and the set $S(Y)$ is formed by the right and upper sides of the square [without the point (1; 1)]. The set $S(Y)$ is obviously externally stable since each point $y \in Y$ in which $y_1, y_2 < 1$ can be put in correspondence. For example, the point $y_0 = ((y_1 + 1)/2, 1)$ and $y_0 > y$.

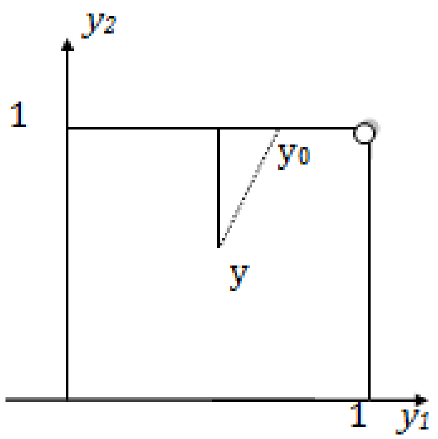


Fig. 3. 2. Graphical representation of a set of Slater-optimal alternatives

The definition of an effective (weakly effective) solution is static in the sense that it is based on a pairwise comparison of acceptable solutions and does not have a commonality with the question of whether it is possible to "smoothly" move from one solution to another, better, at a positive rate, increasing each criterion. Under such conditions the possibility of such a transition in some models is considered very interesting.

An economic model of net exchange, where each consumer participates in the exchange seeking for acquiring a set of goods of the greatest utility, i.e. to maximize its value function, can serve as an example.

This kind of model was explored in the nineteenth century by F. Echart and V. Pareto. What effective in this model is the state (distribution of goods among consumers), which cannot be improved by redistributing goods between exchange participants without "defeating the interests" of some of them.

Thus, Pareto optimality reflects the idea of economic equilibrium: when the state of the economy is not effective, an exchange will occur, which will lead to the transition to an effective state.

3.3 Theoretical and practical value of an effective solution

The concept of an effective solution is a direct generalization of the notion of the maximum point of a numerical function to the case when several functions are considered.

As a rule, in applied problems a set of such alternatives is not empty and moreover, externally stable. Therefore, the optimal solutions should be chosen among the effective alternatives.

However, if for a single-criterion problem any solution can be taken as the optimal one that leads to the criterion reaching its maximum (since the optimal solutions are equivalent), then in a multi-criterion problems a set of effective alternatives is very rich in non-equivalent (and substantially different in content) solutions, as a rule. Therefore, a meaningful choice of the optimal solution is impossible without involving more complete information about the advantages of alternatives.

Nevertheless, the concept of an effective solution plays an important role in the theory of multi-criteria optimization.

Although an effective solution is usually not the only one, but a set of effective alternatives are still much narrower, than the original set of all the solutions. Taking this into account, the construction of a set of effective solutions (or their estimates) is the first step in the implementation of a large number of procedures and methods of multi-criteria optimization.

As in the case of only two or three criteria, a set of effective estimates can be represented graphically, then the analysis of two- and sometimes three-criteria tasks is often more convenient to do by choosing the optimal solution(s) directly from the graph of effective estimates.

For example, this approach forms the basis for a "cost – effectiveness" method.

One alternative of this method encompasses the following:

– each sample is evaluated against two criteria: production cost B and effectiveness of implementation the tasks put E. The values of these criteria are calculated by the methods have been specially applied;

– a graph of estimates is constructed corresponding to all these samples, and the samples are highlighted, among which the optimal one is selected;

– the final selection is done by a decision-maker on the basis of the graph analysis as the graph demonstrates the costs necessary for achieving effectiveness increase.

Example 3.3. There is a need to compare 6 projects against the criteria of cost and efficiency. A graphical representation of project evaluations is shown in Figure 3.3. Since it is desirable the criterion B (cost) to be minimum and the criterion E (efficiency) – maximum, then, the graph demonstrates the advantageous projects 1, 4, 3.

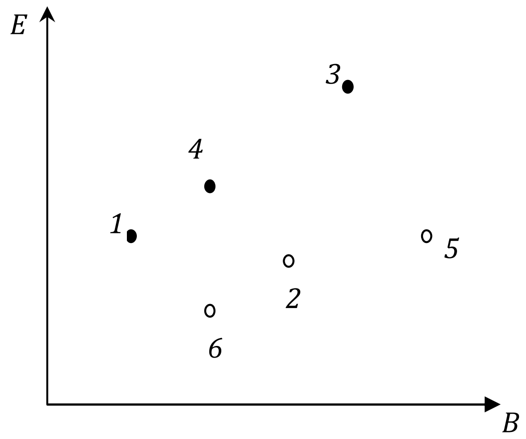


Fig. 3.3. Graph of project evaluations in the "cost – effectiveness method"

Narrowing the choice set to the set of effective solutions (or a subset of them) is important not only for itself, but for the reason that a narrower subset allows to make various assumptions simplifying further analysis. In addition, effective solutions can have interesting and practically important properties that the other solutions lack.

Example 3.4. Suppose there are n industries engaged in the production of n products (items) of consumption. Each industry can produce only one product, through multiple production processes.

Let's denote the set of production processes available to the i-th industry by Λ_i. Consider the set Λ_i is the final one.

If take the total number of labour resources as a unit, then intensity of the i-th industry work can be characterized by the value: $u_i \geq 0$, that shows the share of available labour resources used in this industry. Thus, vector $u = (u_1, u_2, \ldots, u_n)$ describes the performance of all industries. It is clear that with the full use of labour resources, $\sum_{i=1}^{n} u_i = 1$. If the components of the vector u are positive and their sum is equal to one, it is referred to as feasible.

Let the quantity of the j-th product which the i-th industry produces when it operates at a unit intensity level ($u_i = 1$) and the process $\lambda_k \in \Lambda_I$ applied.

We assume that $a_{ij}^{\lambda_k} \leq 0$, when $i \neq j$, but $a_{ii}^{\lambda_k} > 0$. Negative values $a_{ij}^{\lambda_k}$ are interpreted as the amount of materials (products) that are spent in production.

The expressed assumption about signs indicate that every industry can use all kinds of materials, however, it produces only one product.

Let's call vector: $a_{ij}^{\lambda_k} = (a_{i_1}^{\lambda_k}, a_{i_2}^{\lambda_k}, \ldots a_{i_n}^{\lambda_k})$ a i-th industry process vector.

Each production process x_k has its own process vector.

When activities of each sector of the selected production process, and more specifically, if the defined set of production processes are fixed as $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, then the output of pure product j produced by the whole system is $c_j = \sum_{i=1}^{n} u_i a_{ij}^{\lambda_i}$.

We denote a square matrix, where rows are vector processes, $a_i^{\lambda_k}$, $I = 1, 2, \ldots n$, with A^{λ}. Then, the j - th component of the vector: $c = uA^{\lambda}$ is the output of pure product j as a result of a fixed set of processes λ and a feasible vector u.

Let A be the set of matrices A^{λ}, each of which corresponds to a certain set of processes: $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_i \in \Lambda_i$. Vector (the vector estimates): $c = (c_1, c_2 \ldots, c_n)$ is called realized (or achievable), if $c = uA^{\lambda}$ for some matrix A^{λ} and a feasible vector u.

The special attention should be drawn to vectors: $c = (c_1, c_2 \ldots, c_n)$ which have positive components. Indeed, there is an implemented vector c, where $c_j > 0$, $j = 1, 2, \ldots n$ that gives an evidence of possibility to organize the production of all products in such a way that each of the industries will produce a product in a quantity more than there is a need of its consumption by all other industries.

Consider the geometric interpretation of this model.

Each process vector $a_{ij}^{\lambda_k}$ can be represented as a point in space E^n. Matrix A^{λ} corresponds to n of such points (one per each industry).

Vector $c = uA^{\lambda} = \left(\sum_{i=1}^{n} u_i a_{i_1}^{\lambda_1}, ..., \sum_{i=1}^{n} u_i a_{i_n}^{\lambda_i} \right) = \sum_{i=1}^{n} u_i a_i^{\lambda_i}$, i.e. it is a point of convex hull of n process vectors.

Thus, the set of realized vectors c is an association of convex hulls of vectors $a_1^{\lambda_1}, a_2^{\lambda_2}, ..., a_n^{\lambda_n}$ which make up matrices $A^{\lambda} \in A$.

To illustrate this, consider a simple example with numerical data.

Example 3.5. Suppose that $n = 2$ in the above problem;

$$\Lambda_1 = \{1; 2\};\ \ \Lambda_2 = \{1; 2; 3\};$$

$$a_1^{\ 1} = (2; -1);\ \ a_1^{\ 2} = (3/2; -2);$$

$$a_2^{\ 1} = (-1; 1/2);\ \ a_2^{\ 2} = (-2; 3);\ \ a_2^{\ 3} = (-4; 4).$$

The task of determining the best options of production and labour resources is to ensure the largest possible output of all products produced by industries.

Solving

Draw the points which satisfy vector processes $a_i^{\lambda_i}$ on a graph (see Figure 3.4) In our example, they are: $a_1^{\ 1} = (2; -1)$; $a_1^{\ 2} = (3/2; -2)$; $a_2^{\ 1} = (-1; 1/2)$; $a_2^{\ 2} = (-2; 3)$; $a_2^{\ 3} = (-4; 4)$.

The segments connecting these points correspond to the convex hulls of these vectors, i.e. vectors.

The realized vectors c are shown by those segments which are located in the first quarter of the coordinate plane.

It is obvious that the maximum product output will be achieved if $\lambda' = (1; 2)$ that is the first industry uses the first process, and the second

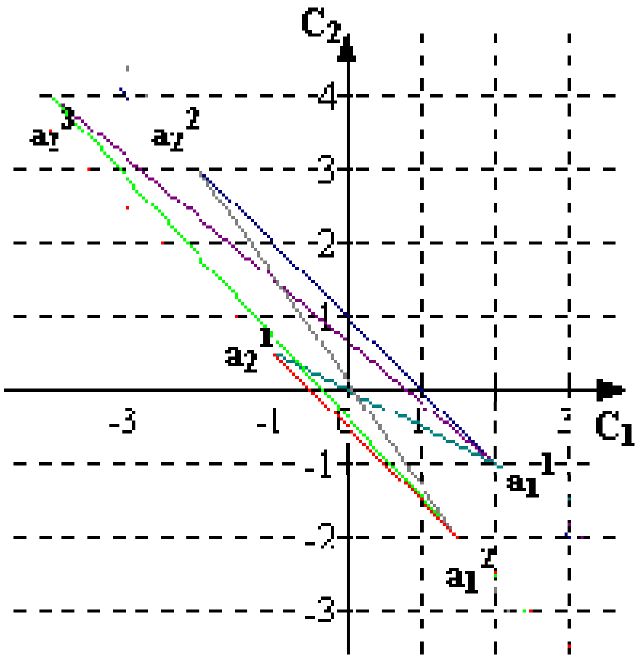


Fig. 3.4. Graphical interpretation of example 3.5

industry – the second. Then any efficient production plan can determine the pair (λ', u), here $u = (u_1, u_2)$, $0{,}5 < u_1 < 0{,}75$, $u_2 = 1 - u_1$. These conditions ensure the feasibility of vector).

Thus, the plan: $x = \{\lambda, u\}$, where λ is production processes, and u is an implemented vector), is characterized by the vector criterion: $c(x) = (c_1(x),\ldots, c_n(x))$, where c_j is the output of pure j-th product.

The plan x^* is called *effective* if there is no feasible vector u and matrix A^* for which $c_j(x) \geq c_j(x^*)$ at least one of these inequalities is strict. Vector $c(x^*)$, which corresponds to the effective plan x^* is also called *effective.*

The structure of the effective vectors with positive components is characterized by the following statement:

If there is a realized vector with positive components, then all efficient vectors with positive components lie in the convex hull of the vectors of the processes $a_{ij}^{\lambda_k}$ *which form the matrix* $A' \in A$, *and each point in this convex hull, located in the positive octant is the effective vector.*

In other words, when there is a valid plan that ensures the excess output of each of the products, there is a certain production process for each industry included in the set λ' which allows you to determine all the effective vectors with positive components only for the sake of redistribution of labour resources.

Thus, any effective plan that provides an excess output of each product has the form of (λ', u), where u is a feasible vector, or it is an equivalent plan of the named type.

3.4 Effective alternatives: properties and the ways of their finding

Consider a problem of multi-criteria optimization:

$$\begin{aligned}
&f_i(x) \to \max,\ i \in I_1,\\
&f_i(x) \to \min,\ i \in I_2,\\
&x \in X,
\end{aligned}$$

and investigate properties of a set of effective alternatives. Let’s formulate them not in respect to the primary vector criterion, but keep in mind a dimensionless vector criterion which consists of monotonic transformations of individual functions, which bring them to a dimensionless form and lead to the problem of minimization. Methods of such a transformation will be discussed below in this paragraph and in paragraph 3.6. So, we have the following initial problem:

$$W_i(x) \to \min,\ i \in I,$$
$$x \in X,$$

where all the functions of $W_i(x) \geq 0$ are reduced to a dimensionless form.

Theorem 3.1. If the set of admissible alternatives X is convex and the objective functions $W_i(f(x)), i \in I,$ are concave in the admissible set X, then for any effective alternative x^* there is such a numerical vector:

$$c = (c_1, c_2, \ldots, c_M),\ \ c_i \geq 0,\ \ \sum_{i \in I} c_i = 1,$$

where the linear criterion, having the following form:

$$F(x) = \sum_{i \in I} c_i W_i(x)\,, \tag{3.2}$$

reaches a minimum in the set X, when $x = x^*$.

Theorem 3.2. Let x^* be an effective alternative to the set of objective functions: $W = \{W_i(x)\}$, $W_i(x) \geq 0$, $i \in I$. Then, there is a numerical vector: $c = (c_1, c_2, \ldots, c_M)$, $c_i \geq 0$, $\sum_{i \in I} c_i = 1$, for which the criterion is of the following form:

$$F(x) = \max_{i \in I} c_i W_i(x)\,, \tag{3.3}$$

reaches a minimum in the set of valid alternatives X if $x = x^*$.

For the component c_i, you can, for example, take the numbers $\frac{\lambda_i}{\lambda}$, here

$$\lambda = \sum_{i \in I} \lambda_i\,,\ \ \lambda_i = \frac{1}{w_i(x^*)}\,.$$

Theorem 3.3. If x^* is an effective alternative to the set of objective functions f, then for any index $l \in I_1$

$$f_l(x^*) = \max_x f_l(x),$$
$$f_i(x^*) \geq f_i(x),\ \forall i \in I_1,\ i \neq l,$$
$$f_i(x^*) \leq f_i(x),\ \forall i \in I_2,$$

or for any index $l \in I_2$

$$f_l\left(x^*\right)=\min_x f_l(x),$$
$$f_i\left(x^*\right)\geq f_i(x),\ \forall i\in I_1,$$
$$f_i\left(x^*\right)\leq f_i(x),\ \forall i\in I_2, i\neq l.$$

In connection with the described properties, three ways of determining effective alternatives can be constructed.

Consider these methods.

The first method (*based on Theorem 3.1*)

The search for the whole set of effective alternatives X^* is reduced to solving of such parametric programming problem:

$$\min_{x\in X}\sum_{i\in I} c_i W_i(x), \tag{3.4}$$

$$\gamma_i\in\Gamma^+=\left\{\gamma_i:\gamma_i\geq 0, \forall i\in I, \sum_{\mathrm{i\in I}}\gamma_i=1\right\}, \tag{3.5}$$

where for all indices $i\in I$ the functions $W_i(x)$ are concave and continuous, and the zone of admissible alternatives X is a convex closed set.

In the case where the functions are not concave or the set of valid alternatives is not convex, the problem (3.4), (3.5) does not allow to find the whole set of alternatives.

Example 3.6. Let the scope of valid alternatives be given by constraints of the following form:

$$X=\left\{x_i\geq 0, i=1{,}2; 0{,}5\leq x_1\leq 3; 0{,}5\leq x_2\leq 5; x_2\geq 4-x_1^2\right\}$$

and the objective function

$$x_1\to\min,$$
$$x_2\to\min.$$

In this problem, set of the effective alternatives are the AB arc (see Figure 3.5). However, since the region is not convex, in the result of minimizing the criterion $F(x)=\gamma_1 x_1+\gamma_2 x_2$ on the set X, $\forall\ \gamma_1, \gamma_2\geq 0, \gamma_1+\gamma_2=1$, no more than two effective alternatives will be found.

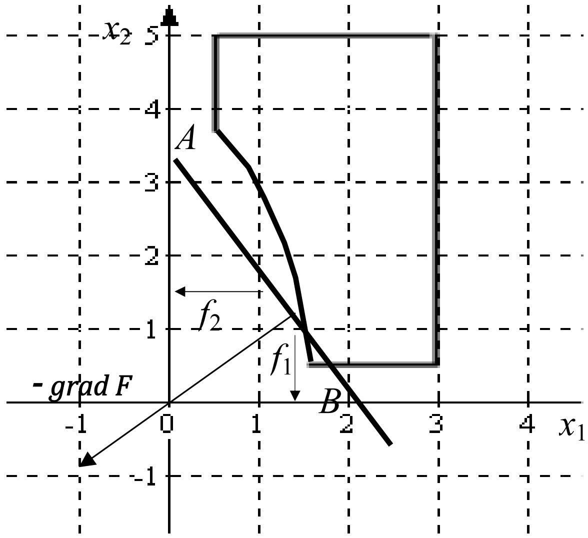


Fig. 3.5. Graphical illustration of example 3.6

The second method (based on Theorem 3.2)

The search for the whole set of alternatives X^* is reduced to solving such parametric programming problem:

$$\min_{x \in X} \max_{i \in I} \gamma_i W_i(x), \tag{3.6}$$

$$\gamma_i \in \Gamma^+ = \left\{ \gamma_i : \gamma_i \geq 0, \forall i \in I, \sum_{i \in I} \gamma_i = 1 \right\}, \tag{3.7}$$

where $W_i(x)$ is monotonic transformations of the objective function $f_i(x)$.

In this case, the requirements of concavity and continuity of the objective functions as well as the convexity of the set of acceptable alternatives are not put forward, but it should be noted that in the case of a lot of solutions to the problem (3.6), (3.7) not all alternatives found can be effective.

Example 3.7. Imagine you are given a discrete set of alternatives: $X = \{ x_j,\ j = 1, \ldots, 5 \}$. The values of the objective functions $w_1(x)$, $w_2(x)$ are given in Table. 3.2. Both functions are minimized.

Table 3.2

The values of the objective functions $w_1(x)$, $w_2(x)$

x_j \ w_i	$w_1(x)$	$w_2(x)$
x_1	2	5
x_2	4	3
x_3	4	2
x_4	5	2
x_5	6	1

Then, when $\gamma_1 = \gamma_2 = 0{,}5$, the minimum criterion (3.6) is achieved in alternatives x_2, x_3 that is the solution will not be the only one. However, it is obvious that $x_3 > x_2$, since x_3 has the best value of the second function, i.e. only the x_3 variant is effective.

Unlike the first method, this method is more general (there are less requirements for objective functions and many alternatives), but when the problem (3.6), (3.7) has several solutions, additional analysis may be necessary.

Note that for the various monotone transformations W_i for the same values of the parameters various effective alternatives will be found. But, if we consider all the values of the parameters $\gamma_i \in \Gamma^+$, the resulting set of effective alternatives will be the same.

Example 3.8. Let the set of acceptable alternatives be given:

$$X = \{x_i \ge 0, i = 1{,}2; x_1 + x_2 \le 1000, 5x_1 + 3x_2 \ge 3500\}$$

(it is shown in Figure 3.6) and the following objective functions:

$$f_1(x) = 37{,}5x_1 + 30x_2 \to \max,$$

$$f_2(x) = 50x_1 + 25x_2 \to \min.$$

Consider two transformations:

$$w^1 = \left\{ \frac{f_1^{\max} - f_1(x)}{f_1^{\max} - f_1^{\min}};\ \frac{f_2(x) - f_2^{\min}(x)}{f_2^{\max} - f_2^{\min}} \right\} \text{ and } w^2 = \left\{ \frac{f_1^{\max} - f_1(x)}{f_1^{\max}};\ \frac{f_2(x) - f_2^{\min}(x)}{f_2^{\min}} \right\},$$

where $f_i^{\max}, f_i^{\min}$ is the maximum and minimum values of the f_i functions, respectively.

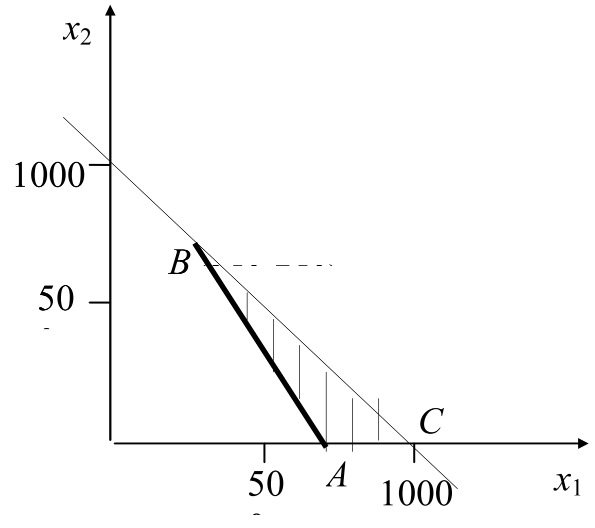


Fig. 3.6. Graphical interpretation of example 3.8

Find the maximum and minimum values of the functions f_1 and f_2 in the set of constraints, namely:

$f_1^{\max}$ is attained at the point (1000,0), $f_1^{\max}$ = 37500,

$f_1^{\min}$ is attained at the point (700,0),

$f_1^{\min}$= 26250,

$f_2^{\max}$ at the point (1000,0),

$f_2^{\max}$= 50000,

$f_2^{\min}$ at the point (250, 750), $f_2^{\min}$= 31250.

We construct the object functions according to the transformations w^1 and w^2, that is

$$w^1 = \left\{ \frac{37500 - 37{,}5x1 - 30x2}{11250};\ \frac{50x1 + 25x2 - 31250}{18750} \right\},$$

$$w^2 = \left\{ \frac{37500 - 37{,}5x1 - 30x2}{37500};\ \frac{50x1 + 25x2 - 31250}{31{,}250} \right\},$$

and consider the problem of parametric programming:

$$\min_{\substack{x \in X \\ \gamma \in \Gamma^+}} \sum_{i=1,2} \gamma_i w_i^1(x) \tag{3.8}$$

and

$$\min_{\substack{x \in X \\ \gamma \in \Gamma^+}} \sum_{i=1,2} \gamma_i w_i^2(x). \tag{3.9}$$

Thus, if $\gamma_1 = 0{,}8$, $\gamma_2 = 0{,}2$, the minimum criterion w^1 (task 3.8) is achieved at point *C*, and the minimum criterion w^2 (task 3.9) – at all points of the *BC* edge (see Figure 3.6).

When $\gamma_1 = 2/3$, $\gamma_2 = 1/3$, the minimum criterion w^1 (task 3.9) is achieved at all points of the *BC* edge, and the minimum criterion w^2 (task 3.6) is at point *B* (see Figure 3.6).

The third method (based on Theorem 3.3)

A set of effective alternatives for objective functions *f* can be found by solving such a parametric programming problem with respect to parameters $z \in Z^{M-1}$:

$$\max_x f_l(x),$$

$$\text{w} \begin{array}{l} f_i(x) \geq z_i,\ \forall i \in I_1,\ i \neq l, \\ f_i(x) \leq z_i,\ \forall i \in I_2, \\ x \in X, \end{array}$$

where Z^{M-1} – $(M-1)$ is dimensional parallelepiped;

$$Z^{M-1} = \prod_{\substack{i \in I_1 \\ i \neq l}} [f_{i(\min)}, f_i^{opt}] \times \prod_{i \in I_2} [f_i^{opt}, f_{i(\max)}],$$

where f_i^{opt} is the optimal value of the corresponding objective function, $f_{i(\min)}$ is the smallest value of the objective function if it is maximized, and $f_{i(\max)}$ is the largest value of the objective function if it is minimized.

Note that it is necessary to choose such a objective function for the main optimized function, the optimum of which is achieved only at effective points

As in the second case, not all alternatives obtained by this method can be effective, and therefore, there is a need for additional analysis.

3.5 General problem of finding compromise solutions

After constructing many effective alternatives to X^*, the group of experts is given the right to choose the best solution in some sense. They give their recommendations to the DM. It either makes a choice of one of their proposed solutions or takes their average result.

Selecting the only one decision from a variety of effective solutions is rather difficult task, because it is possible that the alternative, not optimal against any of the criteria, will be the best in a particular situation of decision-making.

Consider the possible principles of compromise which are applied when selecting a solution from a variety of effective alternatives. Here we will assume that a normal task without priorities is considered, i.e. the criteria are equivalent and normalized. We will also assume that all the criteria are maximized in the set of acceptable alternatives and take only positive values.

3.5.1 Principles of uniformity

In the case where the criteria are normalized and of the same in importance, it is natural to strive to uniformly and harmoniously improve the quality of all partial (local) criteria. This is the essence of the *principle of uniformity*, though at the same time the principle can be implemented in different ways. Consider some of them, assuming that the alternatives are evaluated by n criteria: $y_1, y_2, \dots y_n$ and all the criteria are maximized.

Principle of equality. According to this principle, maximization is carried out provided that the levels of all criteria are the same. In other words, an effective alternative is chosen for which the value of all the criteria are equal that is $y_1 = y_2 = \dots = y_n$. However, this principle is overly rigid. It can predetermine the choice of alternatives that lie outside the area of compromise and even do not give solutions at all, especially when it comes to discrete problems. Examples of such situations, provided that the two criteria y_1, y_2maximized, are shown in Figure 3.7.

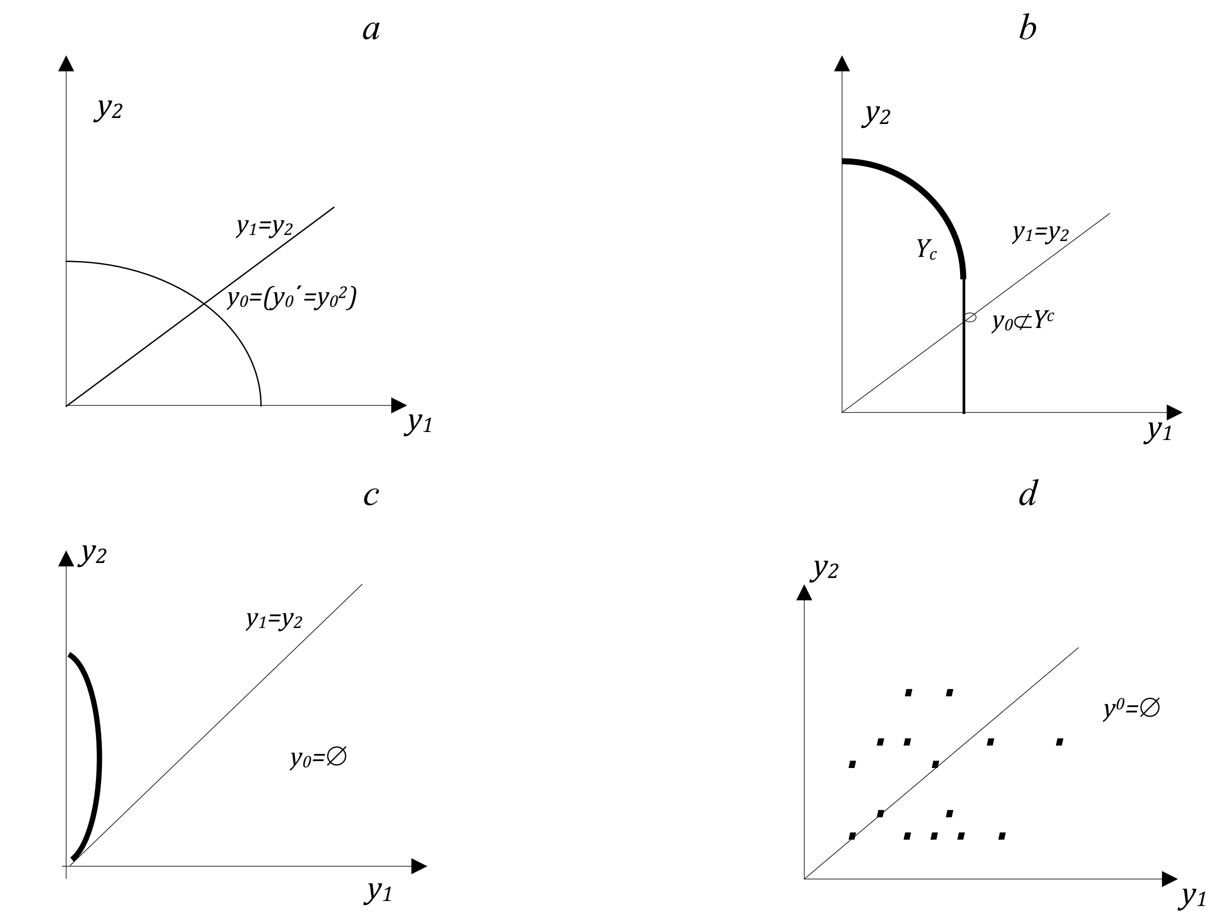


Fig. 3.7. Graphical interpretation of situations of decision-making based on the principle of equality: *a* – existing effective solution; *b* – the solution is beyond the scope of compromise; *c* – no solutions (continuous case); *d* – no solutions (discrete case)

The principle of uniformity (maximin). This principle assumes a uniform increase of levels of all the criteria by "pulling" the worst of them, i.e. the criterion with the lowest level. In addition to uniformity, this principle has another important essence which is to ensure a guaranteed level of the minimum criterion min y_j. It is often called the principle maximin (or minimax in the problem of minimization).

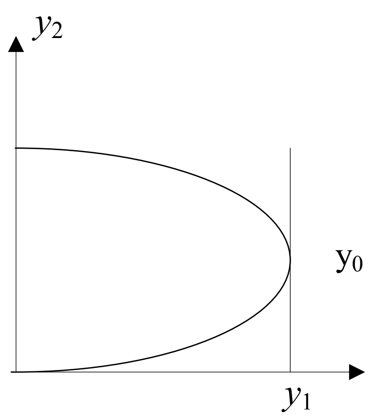


Fig. 3.8. Illustration of the use of the principle of uniformity (maximin)

This principle is illustrated in Figure 3.8 in the condition of two criteria. Both criteria are maximized here. Effective alternatives will be located on the so-called North-Eastern border of the set of acceptable solutions. According to the principle of uniformity, it is necessary to choose a solution that provides the maximum value of the criterion with the lowest level. In this case, the criterion y_1. Therefore, it is rational to choose such a solution: $y_0 = \max \min y_1$.

The principle of the best uniformity. In this case, there is some strengthening of the idea of uniformity in comparison with the previous model, namely: if several solutions are obtained as a result of applying the maximin criterion, then the second minimum is determined, and its maximization is carried out (Figure 3.9).

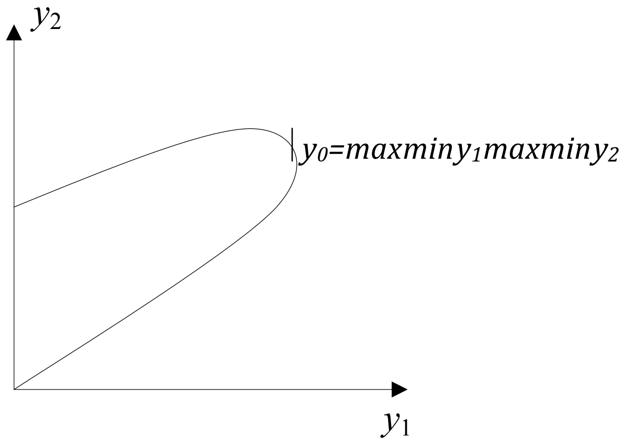


Fig. 3.9. Graphical interpretation using the principle of the best uniformity

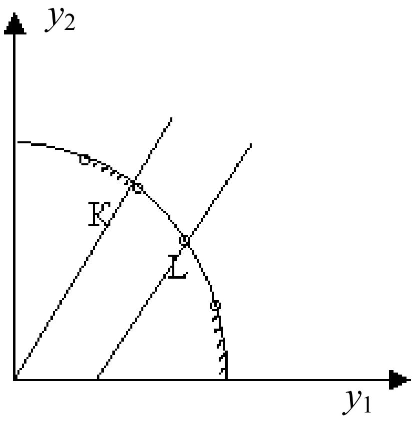


Fig. 3.10. Graphical interpretation using the principle of quasi-equality, $KL = Y^0 = \{y\colon |y_1 - y_2| \le \delta\}$

The principle of quasi-equality. In this case, all local criteria are maximized, provided that their level does not differ by more than a given value δ, that is, there is some weakening of the overlaid rigid principle of equality (see Figure 3.10)

The idea of applying the principles of uniformity is very attractive. The harmonious improvement of the quality of all the criteria is the ideal for optimization. However, often even a slight deviation from these principles can increase significantly the level of one or more criteria, and thus improve the solution of the problem.

The principle of alignment of quality. This principle is based on theorems on the average values of higher degrees. Mathematically, this model is written as follows:

$$\min_{y \in Y^c} \sum_{j=1}^{m} y_j^{-S},$$

here, $S \in (1; S^*)$, $S^* = (\log m)/\log(1+\varepsilon)$.

As the value of the parameter S increases, starting from one, the values of the criteria are aligned, and when $S > S^*$ we reach an ideal alignment equivalent to the model of a sequential maximin.

3.5.2 Principles of fair assignment

The principle of absolute assignment. It provides that a compromise is considered fair when the total absolute level of reduction of one or more criteria does not exceed the total absolute level of increase of other criteria, i.e.

$$\underset{y \in Y^c}{\text{opt}}\, y \equiv \left\{ y : \sum_{j \in I^+} \left|\Delta y_j\right| \geq \sum_{j \in I^-} \left|\Delta y_j\right| \right\} \cap Y^c,$$

where I^+ is a subset of local criteria whose level is improved ($\Delta y_j > 0$); I^- – a subset of local criteria whose level is degraded ($\Delta y_j < 0$), and Δy_j – the value of the increase (or decrease) of the criterion during the transition from the solution y' to y, . $\Delta y_j = \Delta y_j(y', y)$.

Example 3.9. Consider the case where there are two criteria y_1, y_2. Suppose that they are both maximized. Compare the outcome $y\,(2; 4)$ and y′ (3; 1 (see Figure 3.11).

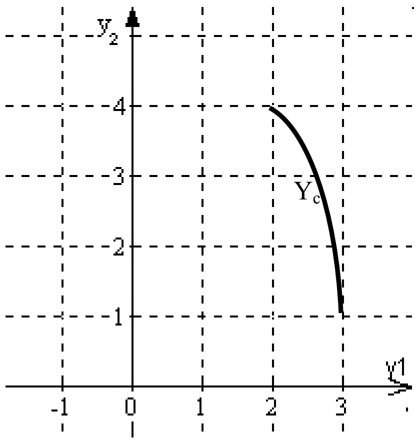


Fig. 3.11. Graphical interpretation of the principle of absolute assignment (Example 3.9)

In the transition from the y' solution to y, the criterion y_1 is reduced to such a value:

$$|\Delta y_1| = |\, y_1 - y'_1| = 3 - 2 = 1,$$

The criterion y_2 is increased by the following:

$$|\Delta y_2| = |y_2 - y'_2| = |4 - 1| = 3.$$

In other words, in the transition from solution y' to solution y, we make a concession by the first criterion: $\Delta y_1 = 1$, but from the position of the second criterion we will win a larger value, that is $\Delta y_2 = 3$ (see fFg. 3.11).

So, since the absolute increase in criteria Δy_2 exceeds the absolute decrease in Δy_1, then the solution y' is considered to be better than y ($y' > y$).

The model of criteria sum maximization (model of integral efficiency) corresponds to this principle, namely:

$$\operatorname{opt} y \equiv \max_{y \in Y^c} \sum_{j \in I} y_j .$$

A serious disadvantage of the principle of absolute assignment is that it does not exclude a sharp differentiation of the levels of individual criteria, that is, a high value of the generalized criterion can be achieved at the expense of one or a group of criteria at a very low level of others. However, this principle has an important advantage – it is convenient for implementation.

The principle of relative assignment. Suppose that all the local criteria forming the efficiency vector are of equal importance. Then, we will consider to be fair such a compromise, where the summative relative level of decrease in quality of a decision against one or on several criteria does not exceed, i.e. less or is equal to the summative relative level of increase in quality against the other criteria. The corresponding to this principle model is referred to as the model of fair *relative assignment*. It is written in this form:

$$\operatorname*{opt}_{y \in Y^c} y \equiv \left\{ y : \left| \sum_{j \in I^+} \eta_j \right| \geq \left| \sum_{j \in I^-} \eta_j \right| \right\} \cap Y^C ,$$

where η_j is the relative change of the "price of assignment" criterion, which is calculated by the formula:

$$\eta_j \ \eta_j = \frac{\Delta y_j (y', y)}{\max\{y', y\}} .$$

Consider the mathematical interpretation of the described principle.

Example 3.10. Let two alternatives be given in the zone of compromises Y^c: y (2; 7) and y' (3; 5) (Figure 3.12), the quality of which is assessed by the criteria y_1 and y_2 (both of them are maximized). Alternative y is better than alternative y' in criterion y_1, but inferior to it according to the criterion of y_2. It is necessary to compare these two alternatives and select the best one in terms of the principle of fair compromise.

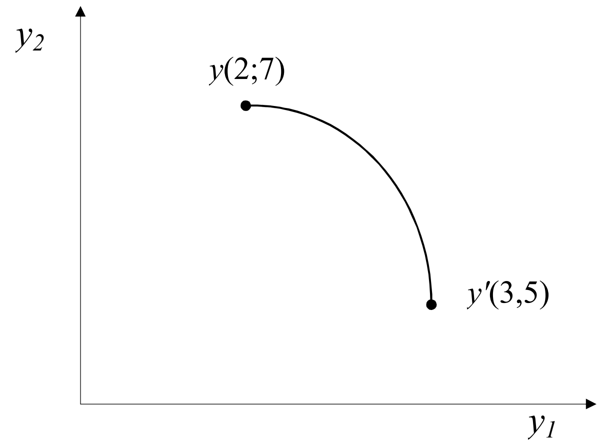


Fig. 3.12. Graphical representation of the relative assignment principle (Example 3.10)

To compare these alternatives, we introduce a measure of the relative decline in the quality of the latter for each of the criteria in the transition from alternative y to y' and vice versa, that is, the price of assignment $\eta_j(y, y')$, namely:

$$\eta_1 = \frac{\Delta y_1(y', y)}{\max\{y_1', y_1\}}, \qquad \eta_2 = \frac{\Delta y_2(y, y')}{\max\{y_2', y_2\}}.$$

Here Δy_1 and Δy_2 are the absolute value of the reduction in the level of criteria in the transition from solution y' to solution y (for criterion y_1) and the reverse transition (for criterion y_2). If the relative magnitude of the reduction of the level criterion y_1 most from the reduction criterion y_2, you should give preference to the solution y. This follows from a comparison of the values of the price of assignment for each of the criteria.

In this example $\Delta y_1 = 1$, $\Delta y_2 = 2$, according to the principle of absolute assignment, solution y is better than solutions y', but on the contrary, according to the principle of relative assignment, the best alternative is y' since $\eta_1 = 1/3$, $\eta_2 = 2/7$, i.e. $\eta_1 > \eta_2$.

Consider the case of continuous change of solutions. Then, the price of the assignment has the form of a logarithmic derivative that is

$$\eta_j(x) = \frac{1}{y}\left|\frac{dy_j(x)}{dx}\right|.$$

Suppose that an increase in the value of the parameter x leads to an increase in the criterion y_1 and a decrease in the criterion y_2. The relative intensity of their change depending on x is characterized by the price of assignment $\eta_1(x)$ and $\eta_2(x)$

(Figure 3.13). As it can be seen from the graphs, when x changes from 0 to x_0, the relative increase in y_1 criterion will be greater than the decrease in y_2 criterion.

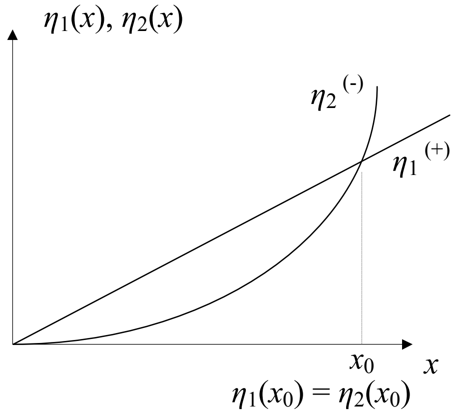


Fig. 3.13. Graphical interpretation of the relative assignment principle (continuous case)

The principle of relative assignment corresponds to a scalar optimization model with a criterion in the form of a product of local criteria, namely:

$$\text{opt}\, y \equiv \max_{y \in Y^c} \prod_{j \in I} y_j .$$

For convenience of calculatig instead of this model it is also possible to use equivalent model of the following type:

$$\text{opt}\ y \equiv \max_{y \in Y^C} \sum_{j \in I} \log y_j ,\quad y_j > 0,\ j \in I.$$

Note that this principle can be applied provided that all local criteria are of equal importance. If this assumption is not valid, then the model must be adjusted, using the weight vector: $\alpha = (\alpha_1, \dots, \alpha_m)$, and find optimal solutions based on such a model:

$$\max_{y \in Y^c} \prod_{j \in I} y_j^{\alpha_j}$$

Here, the principle of fair compromise is somewhat violated, but this is done in accordance with the degree of importance of the criteria and practically results in artificial differentiation of the value of the assignment.

Comparing the two principles of fair concessions examined above, it is possible to draw the following conclusions:

The principle of fair compromise on the basis of assignment has a very clear idea of justice, on the basis of which the best compromise solution is chosen.

The principle of absolute assignment does not depend on the actual value of the criteria and can therefore allow for large differences in their levels, and therefore it should be used only in conjunction with one of the principles of uniformity.

The principle of the relative concessions is particularly sensitive to the criteria, and at the expense of relativity concessions, there is an automatic reduction of its rates as applied to the criteria with higher value and vice versa. As a result, there is a significant smoothing of the criteria levels. It can also be considered that an important advantage of the principle of relative assignment is its invariance to the criteria measurement scale.

If there is an unequal importance of the criteria, the idea of a fair compromise based on the evaluation of concessions loses its clarity as the arguments for the choice of the weighting vector α is rather doubtful, especially when there is a variety of numerous criteria.

3.5.3 Other optimality principles

The principle of the main criterion. According to this principle, one of the local criteria is chosen as the main one and its scalar optimization is carried out, provided that the level of the other criteria is not worse than acceptable, that is

$$\text{opt}\, y \equiv \max_{y \in Y'} y_1 ,$$

where $Y' = \{y \,|\, y_j \geq y_j^{\gamma}, j \in \{2, 3, \ldots m\}\} \cap Y^c$, y_j^{γ} is given the acceptable level of the criterion y_j, $j = 2, 3, \ldots m$.

In this case, the main criterion can be chosen, but it is better to take the one for which it is difficult to determine the permissible level.

Ultimately, with the help of such a model you can implement any compromise scheme and get any optimal solution in the zone of compromises. At the same time, it is often impossible to argue for the selection of an acceptable level of criteria y_j^{γ}.

The principle of maximizing the weighted sum of criteria. It is a modification of the absolute assignment model for the case when the priorities of the criteria are given α_j, $j = 1, 2, \ldots, m$, and is written in this form:

$$\text{opt}\, y = \max_{y \in Y^c} \sum_{j \in I} \alpha_j y_j ,$$

where $\alpha_j \in [0,1]$, $j \in I = \{1, 2, \ldots, m\}$, $\sum_{j \in I} \alpha_j = 1$.

To some extent, this principle has universal significance. For example, when solving the multicriteria problems with the help of this model it is possible to determine a lot of compromise solutions (theorem 3.1).

Note that as in the previous cases it is almost impossible to argue the choice of weight coefficients $\alpha_j J$ for the implementation of any principle of compromise

The principle of maximizing the probability of achieving ideal quality. Often in stochastic vector problems the ideal, desired values of all local criteria y_j^u, and hence the ideal quality is y^u. Then the optimization problem can be formulated in a scalar form with a criterion that means the probability of achieving a complex event $P(y \geq y^u)$:

$$\text{opt } y \equiv \max_{y \in Y^c} P(y \geq y^u).$$

In practice, the methods of calculating the probability of events, even if there are two or three, are very complex. That is why this method can be used only in some specific cases, when the number of events $m \leq 3$, and the calculation of the probability $P(y \geq y^u)$ is quite simple.

3.6 Criteria normalization methods

In real problems, the scales of measurement criteria are often different, and the majority of the models used are sensitive to this fact (invariance is a model of relative concessions). They have sense only in the normalized critical space, and therefore there is a need to perform *normalization* of criteria, i.e. artificially reduce them to a single measure.

The basis for many normalization methods is introduction of the concept of "ideal quality", that is, a vector that has an ideal efficiency value: $y^{ideal} = (y_1^{ideal}, \ldots, y_m^{ideal})$. Then, the choice of the optimal solution becomes equivalent to the best approximation to this ideal vector. Different methods of normalization are obtained depending on what is considered to be an ideal vector and what sense in the "best approximation" is understood.

Often, instead of the actual value of the criteria their deviation from the ideal value: $\Delta y_j = y_j^{ideal} - y_j$, $j = 1, 2, \ldots m$ or dimensionless value of the criterion: $\bar{y}_j = \dfrac{y_j}{y_j^u}$ are considered, obviously $\bar{y}_j \in [0; 1]$, $j = 1, 2, \ldots m$.

When solving multi-criteria optimization problems both methods of scale transformation are used. However, only the second one can be used for normalization since it does not depend on the scale of criteria measurement, does not detract from the significance of any of them and reduces all criteria to a single scale $[0;1]$.

Consider the main ways of normalization from the perspective of a method for choosing the ideal vector y^{ideal}.

Method 1. The specified values of criteria: $y^* = (y_1^*, y_2^*, \ldots, y_m^*)$ are taken for the ideal vector quality, $y^{ideal} = y^*$.

This case is quite rare, because the definition of a given value of criteria is usually associated with serious difficulties, and its reasoning is very biased that leads to the subjectivity of the optimal solution.

Method 2. The vector with components, which are the optimal values of local criteria, is taken as an ideal vector of efficiency. For example, in a problem where all criteria are maximized, the ideal vector should be chosen in this form:

$$y^{ideal} = (\max_{y \in Y} y_1, \max_{y \in Y} y_2, \ldots, \max_{y \in Y} y_m).$$

And then, instead of the absolute value of the criteria, their relative dimensionless value is introduced, that is

$$\overline{y}_j = \frac{y_j}{\max\limits_{y \in Y} y_j}, \ j = 1, 2, \ldots m.$$

The disadvantage of this method of normalization is that it depends significantly on the maximum possible level of criteria, which is determined by the conditions of the problem. In such circumstances, preference is automatically given to the criterion with the highest value of the local optimum, and the equality of criteria is violated.

The same drawback can be found in Savage method (*the principle of the least harm*). Here the ideal vector has the same form, but the space of criteria is transformed into the space of deviations, namely: $\Delta y_j = \max\limits_{y \in Y} y_j - y_j$, $j = 1, 2, \ldots m$, and further choice is based on the principle of minimax. This method also depends significantly on the scale of measurement criteria.

Method 3. When using this method, the components of an ideal vector are the exact upper faces (sup) [or for minimization problems-exact lower faces (inf)] of local criteria that are defined in the solution space *Y*, that is

$$y^{ideal} = (\sup_{y \in Y} y_1, \sup_{y \in Y} y_2, \ldots, \sup_{y \in Y} y_m).$$

Relative criteria are determined by the following formulas:

$$\bar{y}_j = \frac{y_j}{\sup\limits_{y \in Y} y_j}, \ j = 1, 2, \ldots m.$$

This method of normalization is the fairest because it does not violate the equality of any of the criteria, moreover, it is objective and does not depend on the scale of their measurement. Despite this, its application is often impossible since the limit of the criteria is infinity. However, under such conditions, it is possible to approximate the implementation of this method by specifying a certain, sufficiently high level of criteria.

Method 4. Here y^{ideal} components are the maximum possible deviations of a criteria, taking into account the conditions of the original problem, namely:

$$\bar{y}_j = \frac{y_j}{\max\limits_{y \in Y} y_j - \min\limits_{y \in Y} y_j}, \ j = 1, 2, \ldots m,$$

or, when the problem is considered without constraints,

$$\bar{y}_j = \frac{y_j}{\sup\limits_{y \in Y} y_j - \inf\limits_{y \in Y} y_j}, \ j = 1, 2, \ldots m.$$

Note that this method of normalization requires a special check of the invariance conditions with respect to the origin and scale of the criteria measurement, at least with respect to the use of some principles of compromise.

Method 5. Within its framework, normalization is carried out on a single hypercube in this way:

Consider that $\min\limits_{y \in Y} y_j = 0, \ \max\limits_{y \in Y} y_j = 1, \ j \in I,$

or $\inf\limits_{y \in Y} y_j = 0, \ \sup\limits_{y \in Y} y_j = 1, \ j \in I.$

It is also possible to violate the invariance conditions with respect to the origin and scale of criteria measurement for a number of models.

As you can see, the successful solution to the problem of normalization depends greatly on how accurately and objectively it is possible to determine the ideal quality of the solution. Normalization, in fact, is reduced to a certain transformation of the criteria space, i.e. to the choice of a convenient and "fair" topology, in which the task of choosing an alternative for several criteria acquires a strict and understandable sense.

Therefore, the transformation must meet the following requirements:

– to consider the need to minimize deviations from the optimal values for each objective function;

– to have the common beginning of calculation and one and the same order of value change across the whole set of admissible alternatives;

– to maintain a preference relationship across the set of alternatives that are comparable to the original objective functions.

The most widely spread methods rooted in the described above ones are given in the following transformations:

$$w_i^1(f_i(x)) = \begin{cases} \dfrac{f_i^{\max} - f_i(x)}{f_1^{\max} - f_i^{\min}}, \forall i \in I_1, \\ \dfrac{f_i(x) - f_i^{\min}}{f_1^{\max} - f_i^{\min}}, \forall i \in I_2; \end{cases} \tag{3.10}$$

$$w_i^2(f_i(x)) = \begin{cases} \dfrac{f_i^{\max} - f_i(x)}{f_1^{\max}}, \forall i \in I_1, \\ \dfrac{f_i(x) - f_i^{\min}}{f_i^{\max}}, \forall i \in I_2, \end{cases} \tag{3.11}$$

where $f_i^{\max}$ is the maximum, $f_i^{\min}$ is the minimum value of criterion $f_i(x)$ in the set of admissible alternatives X, $\forall i \in I_1 \cup I_2$.

Note that the considered normalization methods are described provided that the criteria are of equal importance, but in most cases, they are not equal, and therefore, it is necessary to take into account their priorities

3.7 Methods of accounting for criteria priority

All methods of criteria priority can be divided into two groups. Consider each of them in more detail.

3.7.1 Hard priority methods

Hard priority methods are based on the fact that criteria are ranked by importance: $y_1 > y_2 \ldots > y_m$, on the basis of which their sequential optimization is performed.

The principle of consistent optimization based on strict priority is that the level of less important criteria cannot be increased if it causes at least a slight decrease in the level of the more important criterion.

In practice, this means that first they find a local optimum for the most important criterion on the whole set of admissible alternatives *X*, which is fixed as an additional constraint. Then the local optimum of the second most important criterion is looked for, but for a new admissible set X^{01} and so on. Thus, there is a gradual narrowing of the admissible set to a single optimal solution or optimal subset, that is

$$X \supset X^{01} \supset X^{02} \supset \ldots \supset X^{0m} = X^{0},$$

$$X^{0j} = \left\{ x \mid y_j(x) \geq \max_{x \in X^{0(j-1)}} y_j(x) \right\} \cap X^{0(j-1)}.$$

This principle of ordering a vector set is called *lexicographic*.

There are such difficulties in applying this method:

1) if there are groups of equivalent criteria, there is a need for their local ordering/ranking within these groups on the basis of one of the principles of uniformity;

2) this method is unsuitable for the solution of many practical problems, because the maximization of the first criterion gives a single solution and the problem is actually reduced to a scalar, because the non-capital criteria are not taken into account.

At the same time, this principle gives good results when using a quasi-optimal approach, behind which at each stage a quasi-optimization is carried out, that is, not the only optimum is found, but some area close to it, namely:

$$X^{0j} = \left\{ x \middle| y_j(x) \geq \max_{x \in X^{0(j-1)}} y_j(x) - \Delta y_j \right\} \cap X^{0(j-1)}$$

where Δy_j is permissible deviations of the *j*-th criterion from the exact optimum.

At the same time, the level of permissible deviation from the optimum is determined taking into account the importance of the criteria, the accuracy of the problem statement and some practical considerations

Note that this approach at the last stage gives not one optimal solution, but a rather narrow quasi-optimal subset, so the only solution is chosen by the DM.

The advantages of the strict priority method are that when using it there is no need for quantitative characteristics of the importance of the criteria, it is enough only to organize them in terms of significance.

3.7.2 Methods of flexible priority

Such methods require quantification of the priority, which allows only a certain degree of preference to be given to more important criteria when choosing a solution. Quantitative estimates of priorities are set, as a rule, in the form of the following vector:

$$\alpha = \left(\alpha_1, \underset{j}{\alpha_2}, \ \ldots, \alpha_n\right), \ \ \alpha_i \geq 0, \ \ i \in I, \ \sum_{i \in I} \alpha_i = 1.$$

Depending on the compromise method applied, different variations of the methods of calculation of prioritiesare used. For example:

The principle of uniformity with priority. Optimization is carried out in accordance with one of the following requirements:

opt $y = (\alpha_1 y_1 = \alpha_2 y_2 = \ldots = \alpha_n y_n)$ – represents the principle of equality based on priority;

$\text{opt}\, y = \max\limits_{y \in Y^e} \min\limits_{j} \left(\alpha_j y_j\right)$ (the principle of uniformity with priority);

$\text{opt}\, y = \max\limits_{y \in Y^e} \min\limits_{j}{}_1 \left(\alpha_j y_j\right) \max\limits_{y \in Y^e} \min\limits_{j}{}_2 \left(\alpha_j y_j\right) \ldots$ (the principle of the best uniformity with priority).

The principle of equitable assignment with priority. Optimization fulfills this requirement:

$$\text{opt}\, y = \max_{y \in Y^e} \sum_{j \in I} \left(\alpha_j y_j\right) \quad \text{or} \quad \text{opt}\, y = \max_{y \in Y^e} \sum_{j \in I} \left(\alpha_j \log y_j\right).$$

Other optimality principles with priority. The optimization is carried out according to the following rule:

$$\text{opt}\, y = \max_{y \in Y^e} \sum_{j \in I} \left(\alpha_j y_j\right)^{-S}.$$

The advantage of the method of flexible priorities is that they allow reasonable to give preference to more important criteria, given the level of importance.

The disadvantage of these methods is the difficulty of finding numerical priority values. As a rule, such estimates are subjective.

Remark 1. Carrying out the transformation of space using vector α, it is necessary to take into account which principle of optimality will be used to select one of the effective solutions to the problem.

Remark 2. The different importance of the criteria can also be taken into account in normalization. In this case, the normalization is carried out considering the characteristics of the priority, for example, the weight vector, namely:

$$\overline{y}_j = \frac{\alpha_j y_j}{y_j^{ideal}},\ j \in I,$$

where y_j^{ideal}, $j \in I$ is an ideal vector (see subsection 3.6).

However, the accounting for priority is better to proceed after the criteria have been normalized for the reasons of clarity of it argumentation.

3.8 Methods of solving multi-criteria optimization problems

3.8.1 Methods of consolidation to a generalized criterion (convolution)

Firstly, let us consider the examples of the methods of solution which consisting in the consolidation of the initial multi-criteria problem to the scalar type by the way of the formation of some generalized criterion. All of these methods are based on the following scheme:

1. All of criteria are normalized; it means that they are reduced to a comparable dimensionless shape.

2. They are "curtailed" into the one objective function, forming a so-called generalized criterion, in which the relative importance of every of the criteria is taken into account with the help of such weighting factors: $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_i \geq 0$, $i \in I$, $\sum_{i \in I} \alpha_i = 1$.

As a consequence, the initial multi-criteria problem is reduced to a simple optimization problem with one criterion.

The most common types of convolution are:

I. The generalized criteria which are based on the weighted average function:

$$F = \left(\sum_{i=1}^{m} \alpha_i k_i^s \right)^{1/S},$$

where k_i, i = 1, 2, … m are normative local criteria; α_i, i = 1, 2, … m are weighting factors.

Among this group, a generalized criterion of this kind is especially provided:

$$F_{\Sigma} = \sum_{i=1}^{n} \alpha_i k_i \ ,$$

which is a linear convolution of local criteria (named *Weighted Sum*). It is easy to use because it allows keeping the linearity of output function. Therefore, in other words, if the initial criteria are linear, then the resulting criteria will also be linear.

II. The multiplicative convolution $\Phi_{\pi} = \prod_{i=1}^{m} k_i^{\alpha_i}$.

In problems, where the criteria tare minimized and maximized at the same time, the criterion of this kind is often used:

$$F = \frac{\sum_{i \in I_1} f_i(x)}{\sum_{i \in I_2} f_i(x)}.$$

There the sum of the criteria, which are maximized is written in the numerator, and the denominator is the sum of the criteria that are minimized.

The disadvantage of this criterion lies in the fact that it is based on the clear assumption, according to which insufficient level of one indicator may be compensated or offset by the other; for example, the low production of output is apparently offset by the low cost of products.

Now, let us recall the popular "criterion of human judgment". It has the form of a fraction, where the assessment of the person's dignity by other people is written in the numerator, and the denominator is their opinion of themselves.

III. The following criterion is often used too:

$$F(x) = \min_{y \in Y^c} \left(\frac{f_i(x)}{\alpha_i} \right).$$

According to it, the maximin problem with a scalar criterion is considered instead of a multi-criteria problem.

In practice, the method of *purposive programming* has also been widely used.

It is based on the summary of all the criteria in one generalization, which means the distance from this vector estimate to an inaccessible ideal point: $b^* = (b_1^*, \ldots b_m^*)$.

The most commonly used generalized criterion of this kind is given below:

$$F(x)=\sum_{i=1}^{M}\alpha_i(f_i(x)-b_i^*).$$

It allows to find optimal solutions of linear deterministic problems using the simplex method.

Example 3.11. Solve the following multi-criteria optimization problem using the convolution method if the criteria priorities: α_1 = 0,7 and α_2 = 0,3. The criteria are considered normalized.

$$\begin{aligned}
&f_1(x_1,x_2)=3x_1+2x_2\to\max,\\
&f_2(x_1,x_2)=-x_1+2x_2\to\max,\\
&2x_1-x_2\ge -4,\\
&2x_1+3x_2\le 24,\\
&20x_1-5x_2\le 100,\\
&5x_1+20x_2\ge 100,\\
&x_1,x_2\ge 0.
\end{aligned}$$

Solving

Since the criteria of the problem are normalized then it is not necessary to normalize them. Therefore, let us perform the convolution of the criteria taking into account their given priorities and direction of optimization. Since the initial criteria are linear, we will apply a linear convolution, namely:

$$F(x_1,x_2)=\alpha_1 f_1(x_1,x_2)+\alpha_2 f_2(x_1,x_2).$$

Then the resulting criterion according to our data will look like this:

$$F(x_1,x_2)=0,7(3x_1+2x_2)+0,3(-x_1+2x_2)=2,1x_1+1,4x_2-0,3x_1+0,6x_2=$$

$$=1,8x_1+2x_2.$$

on its basis such a scalar problem is formulated:

$$\begin{aligned}
&F(x_1,x_2)=1,8x_1+2x_2\to\max,\\
&2x_1-x_2\ge -4,\\
&2x_1+3x_2\le 24,\\
&20x_1-5x_2\le 100,\\
&5x_1+20x_2\ge 100,\\
&x_1,x_2\ge 0.
\end{aligned}$$

To solve this problem, we can use the simplex method or solve it graphically.

As a result of the decision $x_1 = 6$, $x_2 = 4$, and the value of the integral criterion $F(x_1, x_2) = 26$, the initial criteria is $f_1(x_1, x_2) = 26$, $f_2(x_1, x_2) = 2$.

3.8.2 Method of the main criterion

Let us consider the problem of multi-criteria optimization in which all the criteria are minimized and ordered according to their importance, namely:

$$f_i(x) \to \min, \; i \in I,$$

$$x \in X,$$

$$f_1(x) \geq f_2(x) \geq \ldots \geq f_M(x).$$

The main idea of the method is that the original multi-criteria optimization problem can be replaced by a one-criterion problem with additional constraints, which allow to take into account the requirements described by other criteria in the full sense.

There is a scheme of the method.

1. Choose one main criterion $f_1(x)$ for optimization.

2. For less important criteria $f_2(x), \ldots f_M(x)$ allowable values are calculated $\bar{f}_2, \ldots \bar{f}_M$.

3. The criteria $f_2(x), \ldots f_M(x)$ are replaced by constraints of the following form:

$$f_i(x) \leq \bar{f}_i, \text{when } \; i \in I.$$

4. Instead of the output consider such a scalar problem:

$$\begin{aligned} &f_1(x) \to \min, \\ &f_i(x) \leq \bar{f}_i, \quad i \in I, \\ &x \in X. \end{aligned}$$

The advantage of the described method is that its implementation does not require a quantitative assessment of the priorities of the criteria. However, it also has a disadvantage. It is a complexity of establishing acceptable levels of criteria's values. In most cases, they are chosen subjectively. In this regard, if the criteria are equivalent, then any one of them may be chosen as the main one. But it is better to give preference to the one for which it is difficult to set the allowable values.

It is also worth noting that the solution obtained by this method will always be weakly effective, and then when it is unique it becomes more effective.

The method of the main criterion can also be applied to solving problems in which the criteria are maximized and then the additional constraints become as follows: $f_i(x) \geq \bar{f}_i$, $i \in I$.

Generally, the resulting scalar problem can be written as follows:

$$\begin{aligned}
&f_1(x) \to \text{opt},\\
&f_i(x) \geq \bar{f}_i, \quad i \in I_1,\\
&f_i(x) \leq \bar{f}_i, \quad i \in I_2,\\
&x \in X,
\end{aligned}$$

where I_1 is the set of indices for which the objective functions are maximized; I_2 – the set of indices for which objective functions are minimized.

E x a m p l e 3.12. Solve the multi-criteria optimization problem by the method of the main criterion:

$$\begin{aligned}
&f_1(x_1, x_2) = 2x_1 + 3x_2 \to \max,\\
&f_2(x_1, x_2) = x_1 - 3x_2 \to \min,\\
&f_3(x_1, x_2) = -x_1 + x_2 \to \min,\\
&\begin{cases} x_1 + x_2 \leq 10,\\ 2x_1 + 4x_2 \geq 12,\\ x_2 \leq 6, \end{cases}\\
&x_1, x_2 \geq 0,
\end{aligned}$$

if the priorities of the criteria are given as follows: $f_3 > f_1 > f_2$, and the limit values of the criteria are known: $f_1^* = 14$, $f_2^* = 3$.

Solving

Choose the main criterion of the highest importance. In this case, this is a criterion f_3. For the other two criteria, we set the constraint using known limit values. Since the criterion f_1 needs to be maximized, then the corresponding constraint will have the following form: $f_1(x_1,x_2) \geq 14$, i.e. $2x_1 + 3x_2 \geq 14$. For the criterion f_2 (which is minimized), we formulate the following constraint: $f_2(x_1,x_2) \leq 3$, or in a concrete way: $x_1 - 3x_2 \leq 3$ in view of this, the initial multi-criteria problem is reduced to the following scalar problem:

$$f_1(x_1, x_2) = -x_1 + x_2 \to \min,$$

$$\begin{cases} x_1 + x_2 \le 10, \\ 2x_1 + 4x_2 \ge 12, \\ x_2 \le 6, \\ 2x_1 + 3x_2 \ge 14, \\ x_1 - 3x_2 \le 3, \\ x_1, x_2 \ge 0. \end{cases}$$

Having solved this problem, we got the following result: $x_1 = 8{,}25$, $x_2 = 1{,}75$, as well as the value of the criteria: $f_3 = -6{,}5$, $f_2 = 3$, $f_1 = 21{,}75$.

Let us consider the geometric interpretation of the described solution of the problem.

But first, we should construct the domain of the admissible solutions of the initial problem on the coordinate plane (see Figure 3.14, *a*) which is a polygon *ABCDE* (marked in the Figure 3.14 by a thick line), and draw the objective functions of the initial problem.

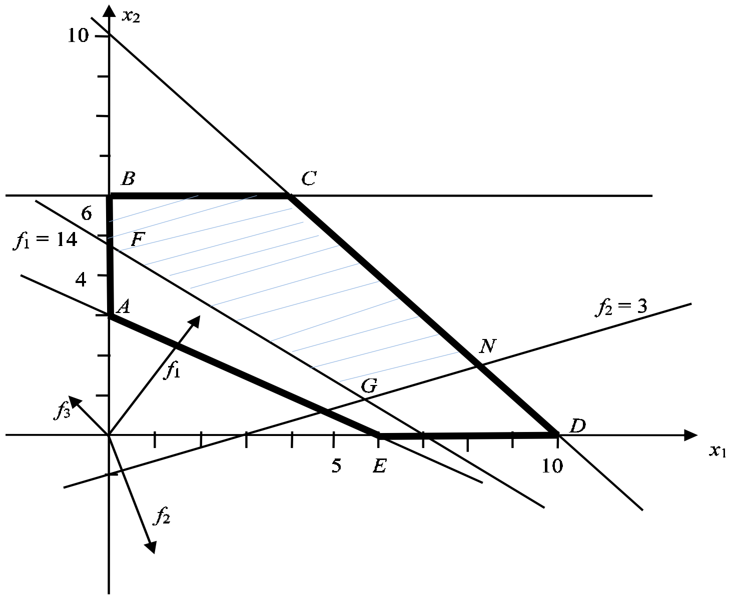
Additional constraints change this area to a set of *FBCNG* (it is shaded in the Figure 3.14). However, all solutions that are not acceptable by the criteria are rejected f_1, f_2, and the solution of the problem is the point *N* with coordinates (8.25; 1.75).

It is easy to see that in this case, the constraint, which belongs to the first criterion, does not affect the solution of the scalar problem, while the constraint that belongs to the second one is active.

Obviously, changing the limit values of the criteria leads to a change in the range of admissible solutions of the scalar problem. For example, the situation when the following acceptable level are: $f_1^* = 10$, $f_2^* = 0$, shown in Figure 3.14, *b*. Here the area of admissible solutions is a polygon of *MLBCNG*. The solution of the problem is now a point *N* with coordinates (7.5; 2.5).

Consequently, each threshold value corresponds to its optimal solution, and it will be weakly effective. Therefore, if change the acceptable level of the criteria, then we can find all the weakly effective solutions to the initial multi-criteria optimization problem.

a



b

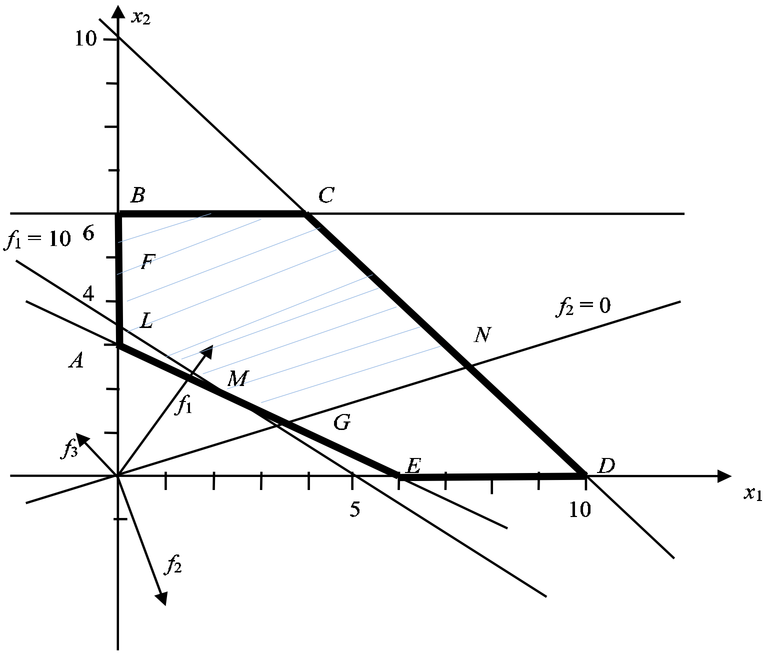


Fig. 3.14. The geometrical interpretation of the solution of the multi-criteria problem to the example 3.12: *a* - acceptable level of the criteria: $f_1^* = 14$, $f_2^* = 3$; *b* - acceptable level of the criteria: $f_1^* = 10$, $f_2^* = 0$

3.8.3. Method of successive concessions

This method as well as the method of the main criterion described above is used in cases, when the criteria are organized according to their importance but quantitative estimates of their priorities are unknown. Let us describe its application in solving a problem of the following form:

$$f_i(x) \to \min, i \in I,$$
$$x \in X,$$
$$f_1(x) \geq f_2(x) \geq \ldots \geq f_M(x).$$

The essence of the method of successive concessions is that the initial multi-criteria problem is replaced by a one-criterion sequence, and the range of admissible solutions is narrowed from problem to problem via additional constraints, which take into account the requirements of the criteria. When formulating each problem that relating to the most important criterion is made a concession, the magnitude of which depends on the requirements of the problem and the optimal solution for this criterion.

Let us describe the scheme of the method.

1. Solve the scalar optimization problem by the most important criterion in the whole set of admissible alternatives *X*, that is,

$$f_1(x) \to \min,$$
$$x \in X.$$

As a result, we get the optimal value of the criterion: $f_1(x)$: $f_1^{\min}$.

2. Solve the optimization problem guided by the following important criterion, taking into account the additional constraint: $f_1(x) \leq f_1^{\min} + \Delta_1$, where Δ_1 is the admissible concession according to the first criterion. This problem can be written as follows:

$$f_2(x) \to \min,$$
$$f_1(x) \leq f_1^{\min} + \Delta_1,$$
$$x \in X.$$

As a result of its solution we get the optimal value of the criterion: $f_2(x)$: $f_2^{\min}$.

Let after the "*k*" steps get the optimal values of the criteria: $f_1^{\min}, f_2^{\min}, \ldots, f_k^{\min}$, then at "(*k* + 1)-th" step solve the following problem:

$$
\begin{aligned}
& f_{k+1}(x) \to \min, \\
& f_1(x) \le f_1^{\min} + \Delta_1 \\
& f_2(x) \le f_2^{\min} + \Delta_2, \\
& \dots\dots\dots\dots\dots\dots \\
& f_k(x) \le f_k^{\min} + \Delta_k, \\
& x \in X,
\end{aligned}
$$

and calculate the optimal criterion value $f_{k+1}^{\min}$.

After considering all the criteria, the problem will be solved. An optimal solution of a multi-criteria problem will be the solution of the last scalar problem.

Therefore, the initial multi-criteria problem is reduced to the successive solving of a range of scalar problems, the number of which will be equal to the number of criteria.

This method gives the opportunity to take into account the priorities of the criteria and avoids increasing their values more than some admissible level (in the case, when the criteria are minimized) or avoids their reduction less than a certain established level (when the criteria are maximized).

The complexity of the method is due to subjectivity in determining the admissible levels. Usually admissible concession is set by experts from the perspective of the optimal value of the criterion and conditions of the problem.

Remark. If the criterion is maximized then the corresponding constraint is formulated as follows: $f_i(x) \ge f_i^{\max} - \Delta_i$, where Δ_i – is the admissible concession of this criterion.

Let us illustrate the use of the method of successive concessions on an example.

Example 3.13. Solve the problem of multi-criteria optimization by the method of successive concessions:

(Consider that criteria are ranked according to importance.)

$$
\begin{aligned}
& f_1(x_1, x_2) = x_1 + 7x_2 \to \max, \\
& f_2(x_1, x_2) = -2x_1 + 3x_2 \to \min x, \\
& f_3(x_1, x_2) = x_1 + x_2 \to \min, \\
& \begin{cases} 3x_1 + 5x_2 \le 15, \\ 2x_1 + 3x_2 \ge 6, \\ x_1 - x_2 \le 1, \end{cases} \\
& x_1, x_2 \ge 0.
\end{aligned}
$$

Solving

Applying the described method, we first solve the optimization problem by the criterion which has the highest priority (in this case, it is the first) in the initial set of admissible alternatives. I tlooks like:

$$f_1(x_1, x_2) = x_1 + 7x_2 \to \max,$$
$$\begin{cases} 3x_1 + 5x_2 \le 15, \\ 2x_1 + 3x_2 \ge 6, \\ x_1 - x_2 \le 1, \end{cases}$$
$$x_1, x_2 \ge 0.$$

The optimal solution for this problem will be the following values: $x_1 = \frac{5}{4}$, $x_2 = \frac{9}{4}$, in this the optimal value of the first criterion $f_1^* = 17$.

Suppose that the experts determined that the concession for the first criterion is $\Delta_1 = 2$. Then the second problem is formulated in accordance with the following rule: optimization is carried out for the next criterion of importance and the one added to the existing constraints, which takes into account the previous criterion. Since the first criterion was maximization, then the constraint would be as follows:

$$f_1(x_1, x_2) = x_1 + 7x \ge f_1^* - \Delta_1,$$

that's

$$x_1 + 7x_2 \ge 17 - 2 = 15.$$

Then, the problem of the second stage is formulated as follows:

$$f_2(x_1, x_2) = -2x_1 + 3x_2 \to \min,$$
$$\begin{cases} 3x_1 + 5x_2 \le 15, \\ 2x_1 + 3x_2 \ge 6, \\ x_1 - x_2 \le 1, \\ x_1 + 7x_2 \ge 15, \end{cases}$$
$$x_1, x_2 \ge 0.$$

The solution of this problem will be pair (x_1, x_2): $x_1 = \frac{15}{8}$, $x_2 = \frac{15}{8}$, and the optimal value of the criterion will be $f_2^* = \frac{15}{8}$.

Now let us do the concession on the second criterion, assuming that the decision of the experts $\Delta_2 = \frac{1}{8}$. Given that the second criterion is minimized, an additional constraint is formulated as follows: $f_2(x_1, x_2) = -2x_1 + 3x_2 \le f_2^* + \Delta_2$, namely:

$$-2x_1 + 3x_2 \le \frac{15}{8} + \frac{1}{8} = 2.$$

Consequently, the third problem takes the following form:

$$f_3(x_1, x_2) = x_1 + x_2 \to \min,$$
$$\begin{cases} 3x_1 + 5x_2 \le 15, \\ 2x_1 + 3x_2 \ge 6, \\ x_1 - x_2 \le 1, \\ x_1 + 7x_2 \ge 15, \\ -2x_1 + 3x_2 \le 2, \\ x_1, x_2 \ge 0. \end{cases}$$

The solution of this problem (and hence the output) will be pair (x_1, x_2): $x_1 = \frac{31}{17}$, $x_2 = \frac{32}{17}$, and the value of the criteria will be $f_1^* = 15$, $f_2^* = 2$, $f_3^* = \frac{63}{17}$.

Thus, having solved three scalar problems, we get the solution of the initial multi-criteria optimization problem. Obviously, varying the magnitude of the concession every time, we have the opportunity to determine other solutions to this problem.

Note that most methods of multi-criteria optimization assume a direct choice of the optimal solution from the set of all available ones. Therefore, it is useful to analyze the results obtained to find out whether they always provide an effective solution, or if not, then specifically predict the possibility of improving it to an effective one.

To illustrate this fact let's consider the following example:

There is an original method of multi-criteria optimization that apply to solving linear problems. It can be described as follows: first, find the optimum points for each criterion separately and then the optimal solution y_0 is obtained in the form of a convex combination of points y^i ($y^{\lambda} = \sum_{i=1}^{m} \lambda_i y^i$, where $\lambda_i \ge 0$, $i = 1, 2, \dots, m$;

$\sum_{i=1}^{m}\lambda_i = 1$), which provides the minimum value of the maximum from the normalized deviations of the criteria f_i from the optimum inherent to them, that is,

$$\max_{i\in M}\frac{y_i^* - f_i(y^0)}{\left|y_i^*\right|} = \min_{\lambda}\max_{i\in M}\frac{y_i - f_i(y^\lambda)}{\left|y_i^*\right|},$$

where $M = \{1, 2, \ldots, m\}$, y_i^* – is the optimal value of the i-th criterion.

Such an identification of the optimal solution has dramatic disadvantages.

Firstly, if some criterion has several points of the optimum on the set X then it is not clear which one should be used since each point y^i corresponds to its solution y_0.

Secondly, the optimal solution obtained in this way, as a rule, will not even be weakly effective.

This situation is illustrated in Figure 3.15. Here, both criteria y_1 and y_2 are maximized, and the area of admissible solutions (polygon *OABCD)* is shown by a thick line. Obviously, the optimal solution of the problem with the criterion y_1 will be the point: $D = (y_1, 0)$, with the criterion y_2 all points on the segment AB will be optimal. If in respect to this criterion we choose the point y_2'' for the optimal solution, then the described method gives a solution to the multi-criteria problem y_0'',, and if we choose the optimal point y_2' –, we get a solution y_0'. However, none of them will be effective since the solution y_0^* is better than any of them.

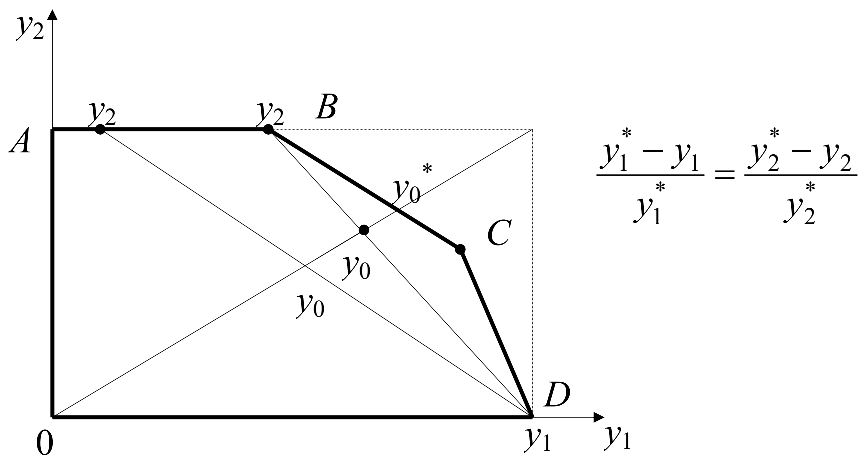


Fig. 3.15. Graphical interpretation of the definition of the solution of the multi-criteria problem

This definition was improved later, namely: the best solution is suggested to consider y^*, which is described by the following condition:

$$\max_{i \in M} \frac{y_i^* - f_i(x^{0*})}{\left|y_i^*\right|} = \min_{x \in X} \max_{i \in M} \frac{y_i^* - f_i(x)}{\left|y_i^*\right|}.$$

3.9 Concept of solving multi-criteria optimization problem with preferences given in set of criteria

Consider the following multi-criteria optimization problem:

$$w_i(x) \to \min, \quad i \in I,$$
$$x \in X,$$

where $0 < w_i(x) < 1$, $i \in I$ and the benefits are given on the set of objective functions W.

Lemma 3.2. For each admissible alternative $x \in X$ with characteristic signs: $0 < w_i(x) < 1$, $\forall i \in I$, in space $W \subset E^M$ there is a vector p and a number $k_0 > 0$, whereas the vector p corresponds to the following relation:

$$p = (p_1, p_2, \dots p_M) = \{ p\colon p_i > 0, \forall\ i \in I, \sum_{i \in I} p_i = 1 \}, \tag{3.12}$$

and the alternative $x \in X$ satisfies simultaneously the following equations:

$$p_i w_i(x) = k_0, \quad i \in I. \tag{3.13}$$

Proof

Since $w_i(x) > 0$, when $i \in I$, then, by dividing the two parts of expression (3.13) into $w_i(x)$, we conclude that

$$p_i = k_0 / w_i(x). \tag{3.14}$$

But since the values of p_i must satisfy the condition (3.12), after substituting the relation: $\sum_{i \in I} p_i = 1$ by expression (3.14), we obtain the following result:

$$k_0 = \frac{1}{\sum_{i \in I} 1/w_i(x)} = \frac{\prod_{i \in I} w_i(x)}{\sum_{q \in I} \prod_{\substack{i \in I \\ i \neq q}} w_i(x)}, \tag{3.15}$$

This proves the lemma.

Remark. The expression (3.15) that indicate the parameter k_0 is a strictly increasing function for each of the variables $w_i(x)$ on the interval (0,1), with $k_0 \in (0; 1/M)$.

Lemma 3.3. If for two non-equivalent alternatives x^* and x^{**} from the set X, the vectors p^* and p^{**} coincide ($p_i^* = p_i^{**}, \quad \forall i \in I$), then $w_i(x^*) = \gamma w_i(x^{**}), \ \forall i \in I$ and $k_0(x^*) = \gamma k_0(x^{**})$, where γ is the constant of proportionality $\gamma \neq 1$.

Proof

The alternative x^* corresponds to the vector p^* that is $p_i^* w_i(x^*) = k_0(x^*)$ for all values $i \in I$ and the alternative x^{**} corresponds to the vector $p_i^{**} w_i(x^{**}) = k_0(x^{**})$, where $\forall i \in I$, hence then, respectively

$$p_i = \frac{\prod\limits_{\substack{j \in I \\ j \neq i}} w_j(x)}{\sum\limits_{q \in I} \prod\limits_{\substack{j \in I \\ j \neq q}} w_j(x)}. \tag{3.16}$$

$$p_i^* = \frac{k_0(x^*)}{w_i(x^*)} \quad \text{and} \quad p_i^{**} = \frac{k_0(x^{**})}{w_i(x^{**})}.$$

Now, taking into account that $p_i^* = p_i^{**}, \ \forall i \in I,$ we get the following result:

$$\frac{w_i(x^*)}{w_i(x^{**})} = \frac{k_0(x^*)}{k_0(x^{**})} = \gamma, \qquad \forall i \in I,$$

this proves the lemma.

Note that the direction which determined by the vector $p \in P^+$ is given for alternatives in the positive octant of the space W of the values of the function w.

The arbitrary vector of weighting factors $p \in P^+$ which satisfies the conditions (3.12) will be interpreted as giving preferences to one objective function over another expressed quantitatively.

Define the direction generated by the vector p in the space W. After assigning this direction to the angles β_i ($i \in I$) between the coordinate axes and the radius vector p, namely:

$$\cos \beta_i = \frac{(w^*, e_i)}{\|w^*\| \cdot \|e_i\|} = \frac{w_i^*}{\sqrt{\sum\limits_{i \in I} w_i^{*2}}}, \qquad \forall i \in I,$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is a basis vector on the axis w_i and $w^* = \{w_i^*\}$ represents a point that located in the space W on the ray p.

Taking into account this relation and the normalization condition, we write down a system of linearly independent equations with the help of which it is easy to find unknown directional cosines, namely:

$$\frac{\cos\beta_i}{\cos\beta_j} = \frac{w_i^*}{w_j^*}, \quad \forall i, j \in I, \quad i \neq j,$$

$$\sum_{i \in I} \cos^2 \beta_i = 1 .$$

On the other hand, the system of equations (3.13) is true for any point w^* via Lemma 3.2, hence

$$\frac{w_i^*}{w_j^*} = \frac{p_j}{p_i}, \qquad \forall i, j \in I, \ i \neq j,$$

and therefore,

$$\begin{cases} \dfrac{\cos\beta_i}{\cos\beta_j} = \dfrac{p_j}{p_i}, & i, j \in I, \ \ i \neq j, \\ \displaystyle\sum_{i \in I} \cos^2 \beta_i = 1. \end{cases}$$

Solving this system, we get the following formula for determining the directional cosines of the vector p:

$$\cos\beta_i = \frac{\prod\limits_{\substack{j \in I \\ j \neq i}} p_j}{\sqrt{\sum\limits_{q \in I} \prod\limits_{j \in I} p_j^{\,2}}}, \quad \forall i \in I . \tag{3.17}$$

Assume that the objective functions are equivalent if $p_i = 1/M$, $\forall i \in I$, then the directional cosines of the vector p in the space W will be determined by the following formulas:

$$\cos\beta_i = \frac{1}{\sqrt{M}}, \quad \forall i \in I .$$

Consequently, assigning the quantitative benefits of the set of objective functions by means of the relation (3.12) shows the direction of the search for solutions in the space W of the selected transformations.

That is why the solution of the vector optimization problem will be assumed as a compromise alternative belonging to the set of effective alternatives and is located in the given direction set by the vector $p \in P^+$ in the space W.

When for a certain alternative x and the given vector $p \in P^+$ the following relation holds: $p_i w_i(x) = k_0$, $\forall\ i \in I$, then we will say that the alternative x lies in the direction determined by the vector $p \in P$.

We will find what value of the parameter k_0 corresponds to an effective alternative, which lies in the given direction that set by the vector p.

Theorem 3.4. If x_0 is an effective alternative for the given vector $p \in P^{+,}$, then it corresponds to the smallest value of the parameter k_0 under which the system of equations (3.13) is performed simultaneously for all values $i \in I$.

If for the transformation $w_i(f_i(x))$, $\forall i \in I$ choose that which looks like (3.10), then, taking into account this theorem, we can give the following definition: *the solution of the vector optimization problem for the given vector of benefits $p \in P^+$ will be assumed as a compromise alternative $x \in X$ that provides the same minimum weighted relative losses: $\widetilde{w}_i(x) = p_i w_i(x)$ against all the criteria simaltaneously.*

3.10 Method of constraints for searching compromise solutions in vector optimization problems

In order to substantiate the computational procedure of searching for the above defined compromise solution, we prove the theorem.

Theorem 3.5. In order the alternative $x^* \in X$ characterized by the following features: $w_i(x^*) > 0$, $\forall i \in I$ be effective in relation to the given vector of benefits $p \in P^+$, it is sufficient it has the only solution of the system of inequalities:

$$p_i w_i(x^*) \le k_0, \quad \forall i \in I, \tag{3.18}$$

with respect to the minimum value of the parameter k_0^* with which this system is compatible.

Proof

Assume the opposite, i.e. the alternative x^* is the only solution of the system (3.18) when the parameter $k_0 = k_0^*$ is not effective. Then, there is an alternative $x' \in X$ that corresponds to the following conditions: $w_i\ (x') \le w_i(x^*)$, $\forall i \in I$, and at least one inequality is executed as strict. Having multiplied these inequalities by $p_i > 0$, $\forall i \in I$, we draw the following conclusion:

$$p_i w_i(x') \le p_i w_i(x^*) \le k_0^*, \forall i \in I,$$

and at least one inequality is strictly executed.

Consequently, we have a contradiction. An alternative x' satisfies the system (3.18) with the value of the parameter k_0 not more than k_0^*. Thereby the theorem is proved.

This theorem brings us to that the previously compromised solution of the multi-criteria optimization problem can be found as the only solution of the system of inequalities (3.18) with a minimal value of the parameter k_0 which this system is compatible with.

In the space of solutions, a compromise alternative corresponds to the point of intersection of the ray where directional cosines are determined by the given vector of benefits $p \in P^+$ with the formulas (3.17) with the zone of effective alternatives.

It follows from the existence of points of intersection that a compromise solution for which the minimum possible weighted losses for all criteria are the same $[p_i w_i(x) = k_{0(\min)}, \forall i \in I]$. If such a point does not exist, then a system of inequalities will be used for a compromise alternative and this alternative will correspond to the point closest to the given ray.

To find a compromise solution we will construct an iterative process with the parameter $k_0 \in (0;\ 1/M)$; at each step of which the compatibility of the system of inequalities (3.15) for $x \in X$ and the given vector p is checked.

The parameter $k_0 \in (0;\ 1/M)$ limits the relative losses: $w_i(x) = w_i(f_i(x))$, $\forall i \in I$. If $k_0 \to 0$, then the relative losses go to 0, i.e. the objective functions $f_i(x)$ go to their optimal values, and when $k_0 \to 1/M$ then the inequalities (3.18) are satisfied in the whole set of admissible alternatives X.

By decreasing the parameter k_0 and thus reducing the weighted loss for all objective functions we are approaching the compromise alternative which ensures minimal losses for all criteria $f_i(x)$.

The iterative process stops, when the smallest value of $k_0(l)$ (l – is the number of the step) in which the system of inequalities (3.18) is compatible in the set of admissible alternatives, differs from the nearest value $k_0(l + 1)$ for which the system is no longer compatible, not more than the value: $\varepsilon \ge 0$. The value ε is given in advance from the considerations of the acceptable time for solving the problem. At the same tim,e if the solution of the system of inequalities is the only one, then this is the desired compromise alternative. If however, it is not the only one, then the relative losses for the obtained alternatives are equivalent to ε. The only compromise alternative can be determined by optimizing any generalized criterion in a set of alternatives equivalent to ε. For example, the linear convolution of the criteria:

$$F(x) = \sum_{i \in I} p_i w_i(x), \tag{3.19}$$

can be minimized in the following set:

$$X' = \{x : p_i w_i(x) \leq k_{0(min)}, \forall i \in I, x \in X\}. \tag{3.20}$$

Such a generalized criterion always allows to find effective solutions.

Let's consider the geometric interpretation of this method.

Example 3.14. Let the equivalent criteria be given and the set of admissible alternatives be linear (see Figure 3.16). Here G is the region of values of the transformed criteria w_1 and w_2 in the set of constraints, Γ is the limit of this set, Ω_j is a part of the range of values of the criteria w_1 and w_2 in which they do not exceed the parameter value.

The criteria w_1 and w_2 are defined by the relations (3.10), that is, they are reduced to dimensionless appearance and minimized.

Since the criteria are equivalent, then $p_1 = p_2 = 1/2$ and $\cos(\beta_1) = \cos(\beta_2) = \frac{1}{\sqrt{2}}$, that is, the direction R is the bisector of the coordinate angle $w_1 0 w_2$. Solutions that provide the minimum relative deviations from optimal values are located in the area G on a ray that goes from the origin to R.

A compromise solution providing the minimum deviations will be the point C^* (the intersection of the ray with the area of effective alternatives). To find it, we will compute successively the least value k_0^l at which the intersection of sets Ω_l and G is not empty.

If the vector found in accordance with the formula (3.16) does not intersect with the area of effective points Ω_l. Then in a region corresponding to the minimum value k_0 at which the system (3.18) is still compatible, a certain set of points will necessarily fall and among them will necessarily be effective, best suited to the benefit given by the weight vector, that is, closest to the ray R.

From these positions, we will analyze some generalized criteria.

Consider the criterion of this form:

$$\min_{x \in X} F(x) = \min_{x \in X} \max_{i \in I} p_i w_i(x), \tag{3.21}$$

where $w_i(x)$, $i \in I$ are described by the relations (3.10).

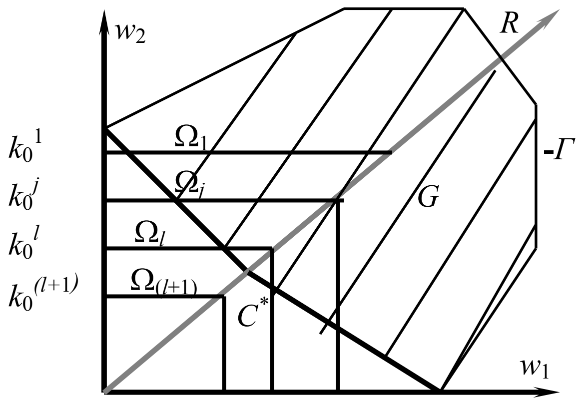


Fig. 3.16. Grafical representation of iterative process of finding a compromise alternative

This method allows us to find an alternative for which either the equation system is executed: $p_i w_i(x^*) = k_0$, $\forall\, i \in I$ and the minimum value of the parameter k_0 or for some values of the equality parameter is not performed and $p_i\, w_i(x^*) < k_{0(\min)}$.

If this alternative is the only one, then it will be a desired compromise solution otherwise, it is necessary to apply an additional criterion (3.19).

Thus, the proposed method which based on the search for additional alternatives to the system of inequalities (3.18) with a minimum value of k_0 can be considered as a way of solving the problem (3.21).

The method of minimizing the criteria that having the form of a convolution (3.19) in the set X does not allow to achieve the above-described solution for the following reasons:

– it essentially depends on the choice of the type of transformation $w_i(x)$ since the different order of the quantities $w_i(x)$ leads to a change in the preferences, and the terms can become comparable in size with small values of p_i;

– when the order of the criteria $w_i(x)$, $i \in I$ is the same, then the benefits may change due to the non-symmetric behaviour of the functions $f_i(x)$;

– if the functions $f_i(x)$ are linear and the admissible set is a polyhedron, then the solution according to criterion (3.19) lies at the vertex of the polyhedron, while the compromise solution will be placed on the edge.

If in the role of a possible transformation we take formulas (3.10), then the t problem of finding a single alternative (3.19), (3.20) can be formulated as follows:

Find the solution of such a parametric programming problem in relation to the parameter k_0 for a given vector of benefits p:

$$\min_x \left[F(x) = \sum_{i \in I_1} p_i \frac{f_i^{\max} - f_i(x)}{f_i^{\max} - f_i^{\min}} + \sum_{i \in I_2} p_i \frac{f_i(x) - f_i^{\max}}{f_i^{\max} - f_i^{\min}} \right], \tag{3.22}$$

taking into account such constraints:

$$x \in X,$$
$$f_i(x) \geq f_i^* = \frac{k_0}{p_i}\left(f_i^{\max} - f_i^{\min}\right), \quad \forall i \in I_1, \tag{3.23}$$
$$f_i(x) \leq f_i^* = \frac{k_0}{p_i}\left(f_i^{\max} - f_i^{\min}\right), \quad \forall i \in I_2.$$

The result of solving the problem (3.22), (3.23) with the minimum possible value of the parameter $k_0 \in (0;\ 1/M)$ will be a desired compromise alternative.

This method is called a *constraints method.*

Let us describe its algorithm.

1. Find the minimum possible value of k_0, in which the constraint system (3.23) is compatible.

2. If the solution of the system is unique then it will be the desired solution of the multi-criteria optimization problem.

3. If the solution of this system of inequalities is not the only one then the further choice is made with the criterion (3.22).

Note that this method does not depend on the type of functions $f_i(x)$ and the set of admissible alternatives X. It is only necessary to have effective ways of verifying for the system of inequalities compatibility (3.23).

3.11 Method of constraints in linear programming multi-criteria problem

Let a set of linear objective functions is given:

$$F = \{f_i(x)\}, \quad i \in I,$$

where $f_i(x) = c^i x = c_1^i x_1 + c_2^i x_2 + \ldots + c_n^i x_n$, $i \in I$, with m the first functions are maximized and the rest *(M - m)* are minimized.

For variables: $x = \{x_j\}$, $j = 1, \ldots, n$, the linear constraints are as follows:

$$Ax \leq b,$$

$$x_j \geq 0, \ \ j = 1, 2, \ldots, n.$$

Apply the method of constraints for solving this problem. According to it, at first, we should perform the conversion of the objective functions, thus:

$$w_i(f_i(x)) = \frac{c^i x_i^0 - c^i x}{c^i x_i^0 - c^i x_{i\min}}, \quad \forall i \in I_1,$$

$$w_i(f_i(x)) = \frac{c^i x \ - c^i x_i^0}{c^i x_{i\max} - c^i x_i^0}, \quad \forall i \in I_2,$$

where $x_i^o = \left(x_{i1}^o, x_{i2}^o, \ldots x_{ij}^o, \ldots x_{in}^o\right)$ is the solution that belongs to the set of constraints and optimizes the *i-th* objective function; $x_{i\max} = \left(x_{i1\max}, x_{i2\max}, \ldots x_{ij\max}, \ldots x_{in\max}\right)$, $x_{i\min} = \left(x_{i1\min}, x_{i2\min}, \ldots x_{ij\min}, \ldots x_{in\min}\right)$ – are solutions that provide the minimum and maximum values of the *i-th* criterion, respectively.

A compromise solution will be one for which the weighted relative losses will be the same and minimal, that is, $p_1 w_1(x) = \ldots = p_m w_m(x) = k_{0\min}$.

According to the method of constraints, this solution can be found from the system of inequalities (3.23), which in this case can be written as follows:

$$\begin{aligned}
& c^i x \ge c^i x_i^o - \frac{k_0}{p_i}\left(c^i x_i^o - c^i x_{i\min}\right), \quad \forall i \in I_1, \\
& c^i x \le c^i x_i^o + \frac{k_0}{p_i}\left(c^i x_{i\max} - c^i x_i^o\right), \quad \forall i \in I_2, \\
& Ax \le b, \quad x_i \ge 0, \quad \forall i \in I.
\end{aligned} \tag{3.24}$$

Solving this system (3.24) will be equivalent to solving the linear programming problem formulated below:

$$\min_x \{k_0 = x_{n+1}\}$$

for the following constraints:

$$\begin{aligned}
& d_{11}x_1 + d_{12}x_2 + \ldots + d_{1n}x_n + d_{1n+1}x_{n+1} + d_1 \ge 0, \\
& d_{21}x_1 + d_{22}x_2 + \ldots + d_{2n}x_n + d_{2n+1}x_{n+1} + d_2 \ge 0, \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& d_{M1}x_1 + d_{M2}x_2 + \ldots + d_{Mn}x_n + d_{Mn+1}x_{n+1} + d_M \ge 0, \\
& a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n - b_1 \le 0, \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n - b_2 \le 0, \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& a_{p1}x_1 + a_{p2}x_2 + \ldots + a_{pn}x_n - b_p \le 0, \\
& x_i \ge 0, \quad i = 1, 2, \ldots n,
\end{aligned}$$

where

$$d_{ij} = \begin{cases} p_i c_j^i, & \forall j = \overline{1,n}, \forall i \in I_1, \\ -p_i c_j^i, & \forall j = \overline{1,n}, \forall i \in I_2, \end{cases} \qquad d_{i,n+1} = \begin{cases} \sum\limits_{j=1}^{n} c_j^i (x_{ij}^0 - x_{ij\min}), \forall i \in I_1, \\ \sum\limits_{j} c_j^i (x_{ij\max} - x_{ij}^0), \forall i \in I_2, \end{cases}$$

$$d_i = \begin{cases} -p_i \sum\limits_{j=1}^{n} c_j^i x_{ij}^0, & \forall i \in I_1, \\ p_i \sum\limits_{j=1}^{n} c_j^i x_{ij}^0, & \forall i \in I_2. \end{cases}$$

Conclusions

One of the problems in decision-making is the presence of a large number of criteria that are not always consistent with each other. Such a situation can be described by mathematical models of multi-criteria optimization problems.

The solution to the multi-criteria optimization problem is to be found among the set of effective, i.e. unsolvable alternatives. Since effective alternatives are either equivalent or uncomparable with each other, then some of the principles of compromise must be used to select one of them.

Criteria may have different value and their priorities can be set quantitatively in the form of a priority vector: $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ $\alpha_i \geq 0$, $i \in I$, $\sum\limits_{i \in I} \alpha_i = 1$, or qualitatively: the relation of benefits to the set of objective functions: $f_1(x) \geq f_2(x) \geq \ldots \geq f_M(x)$. Depending on how the priorities of the criteria are given, which principle of compromise is chosen, and which form has the domain of admissible alternatives and objective functions, different methods are used to find the set of effective alternatives and their corresponding methods of solving multi-criteria optimization problems. Brief characteristics of the most common is given below.

The method of the main criterion consists in changing the multi-criteria problem to one-criterion with additional constraints. This method does not require the normalization of the criteria and the quantification of their priorities. However, it is necessary to have information about the acceptable level of non-main criteria.

The methods of the convolution (are based on the introduction of the integral criterion and the subsequent reduction of the original multi-criteria problem to the scalar). They are convenient to use but have several limitations. In particular, these methods involve the normalization of the criteria and quantification of their priorities;

in addition, they can only be applied to the concave functions and the convex set of admissible alternatives.

The method of the successive concession does not require the normalization of the criteria and the quantification of their priorities. The initial multi-criteria problem is replaced by the consistency of scalar problems. The magnitude of the concession according to each criterion is determined by the DMP depending on the size of the optimum and the meaning of the problem.

Since not always the alternatives found as a result of solving the multi-criteria optimization problem will be effective, it is useful to analyze the given results to find out whether an effective solution has been succeeded, and if not, then, specifically predict the possibility of improving it to an effective one.

SELF-STUDY

Questions for assessment and self-assessment

1. Formulate the general statement of the multi-criteria optimization problem.
2. What alternatives are called effective for Pareto? Effective for Slater?
3. What properties of effective alternatives do you know?
4. Formulate and prove the lemma of effective alternatives.
5. Formulate theorems on the properties of effective alternatives.
6. What methods of finding effective alternatives do you know?
7. Why do we need to normalize the criteria when solving multi-criteria problems?
8. What are the ways to normalize the criteria you know?
9. What is the problem of finding compromise solutions?
10. What is the essence of the principles of uniformity in the search for compromise solutions?
11. What the principles of uniformity in the search for compromise solutions do you know?
12. What is the essence of the principles of concession in the search for compromise solutions?
13. What the principles of concession in the search for compromise solutions do you know?
14. Explain the essence of other principles of optimality in the search for compromise solutions.
15. Which of the other principles of optimality in the search for compromise solutions do you know?

16. What are the methods of the convolution in applying to solving multi-criteria problems?

17. Name the stages of the methods of the convolution.

18. What kinds of the convolution do you know?

19. What are the advantages and disadvantages of the methods of the convolution?

20. Is it necessary to normalize the criteria when using the methods of the convolution?

21. Do we need the quantitative values of the criteria benefit when using the methods of the convolution?

22. What is the essence of the method of the main criterion for solving multi-criteria problems?

23. List the advantages and disadvantages of applying the main criterion method.

24. Is it necessary to normalize the criteria when using the method of the main criterion to solve multi-criteria problems?

25. Do we need the quantitative values of the benefits of the criteria when using the method of the main criterion in solving multi-criteria problems?

26. What is the essence of the method of the successive concession of solving multi-criteria problems?

27. What are the advantages and what are the difficulties of using the method of the successive concession to solve multi-criteria problems?

28. Is the normalization of the criteria in case of the consecutive assignment method applied?

29. Do quantitative values of the advantages of criteria are required when using a serial assignment method?

30. Do the methods of the convolution, the successive concession and the main criterion determine the only optimal solution of the multi-criteria problem?

31. Is it possible to find one of the effective solutions of the multi-criteria problem using these methods?

32. What are the criteria priority accounting methods?

33. Name the methods of the strict priority criteria. What is their essence?

34. List the methods of taking account of a flexible priority criteria. What is their essence?

35. What is the solution of the problem of multi-criterion optimization in the given relationship advantages?

36. What is the essence of the method of constraints when searching for compromise solutions of the vector optimization problem?

Hands-on practice

Task A

1. Construct a set of effective alternatives to such a problem of multi-criteria optimization:

$$f_1(x_1, x_2) = x_1 + 7x_2 \to \max,$$
$$f_2(x_1, x_2) = 2x_1 - 3x_2 \to \max,$$
$$\begin{cases} 3x_1 + 5x_2 \le 15, \\ 2x_1 + 3x_2 \ge 6, \\ x_1 - x_2 \le 1, \end{cases}$$
$$x_1, x_2 \ge 0.$$

2. Solve the multi-criteria optimization problem formulated below with the main criterion method, if the benefits of the criteria are given as follows: $f_3 > f_1 > f_2$.

$$f_1(x_1, x_2) = x_1 + 7x_2 \to \max,$$
$$f_2(x_1, x_2) = 2x_1 - 3x_2 \to \max,$$
$$f_3(x_1, x_2) = x_1 + x_2 \to \min,$$
$$\begin{cases} 3x_1 + 5x_2 \le 15, \\ 2x_1 + 3x_2 \ge 6, \\ x_1 - x_2 \le 1, \end{cases}$$
$$x_1, x_2 \ge 0.$$

3. Solve the following multi-criteria optimization problem using the method of the convolution, if the benefits of the criteria are equal to 0.3; 0.2; 0.5 respectively.

$$f_1(x_1, x_2) = 2x_1 + 3x_2 \to \min,$$
$$f_2(x_1, x_2) = x_1 - 3x_2 \to \max,$$
$$f_3(x_1, x_2) = -x_1 + x_2 \to \min,$$
$$\begin{cases} x_1 + x_2 \le 10, \\ 2x_1 + 3x_2 \ge 12, \\ x_2 \le 6, \end{cases}$$
$$x_1, x_2 \ge 0.$$

4. There are strict priorities set in the set of criteria: $f_3 > f_1 > f_2$. What methods of multi-criteria optimization can be applied? Solve this problem of multi-criteria optimization under such conditions:

$$f_1(x_1, x_2) = -x_1 + 3x_2 \to \min,$$
$$f_2(x_1, x_2) = 4x_1 + x_2 \to \max,$$
$$f_3(x_1, x_2) = x_1 + x_2 \to \min,$$
$$\begin{cases} x_2 \le 8, \\ 3x_1 + 5x_2 \ge 15, \\ x_1 + x_2 \le 10, \end{cases}$$
$$x_1, x_2 \ge 0.$$

5. Let in the set of alternatives: $X = \{x_1, x_2, \ldots, x_{10}\}$, set five criteria, where the criteria f_1, f_4, f_5 are maximized, and f_2, f_3 – are minimized. The values of the criteria on the set X are given in Table. 3.3. Identify the set of effective alternatives.

Table 3.3

	f_1	f_2	f_3	f_4	f_5
x_1	1	3	3	6	2
x_2	2	5	2	7	3
x_3	6	3	5	5	4
x_4	3	2	3	6	5
x_5	9	7	8	5	4
x_6	3	4	5	2	4
x_7	6	4	5	7	5
x_8	3	2	1	4	2
x_9	5	7	4	3	4
x_{10}	7	4	2	5	6

Task B

Formulate mathematical models of the following multi-criteria optimization problems.

1. The enterprise "Morning" has 7 outlets, including shops and a warehouse of production. Daily delivery of goods goes from the warehouse to stores. It is known where stores are located and what the ways of communication there are between them

are. It is necessary to make an optimal route for transportation of products from the warehouse to the stores, taking into account the cost of transportation, the time of delivery, the length of the route, its workload, the quality of roads, provided that the transportation is carried out by one car, and the goods are delivered to each store once a day.

2. In the process of iron ore mining at the Zaporizhzhia Iron Ore Plant a bookmark that hardens is used. It consists of viscous and inert materials. An inert filler for the preparation of a mortar mixture is the waste of energy, metallurgical and mining production, in particular blast furnace slag (x_1), tails of CGOK processing plant (x_2), lime-dolomite material (x_3), sand (x_4) and loam (x_5).

The task is to determine the composition of the mixture so, that its cost is minimal and strength is the maximum. In this case, the following technological conditions must be fulfilled: the water content in the mixture is 20% of the astringent components; the content of cement, limy-dolomite material and sand should be 65, 9, 35, and 18% respectively of the inert components of the mixture.

The dependence of the strength of the mixture on its components is described by the function: $\varphi(x) = 467x_1 + 380x_2 - 54x_3 + 87x_4 - 120x_5 - 23{,}25$.

3. There are three mining sites at mine "Dobropolska". The coal extracted at each of them has different sulfur content, humidity and ash content (see Table 3.4). For each of the districts, the values of the maximum possible and the minimum required amount of extraction are known, as well as the cost of extraction of one ton of raw materials (see Table 3.4). The planned production of the mine at the mine is 3 000 thousand tons. Due to the potential of each site, it is necessary to draw up a plan of mining operations in order to minimize the costs the amount with the output being maximum, and the ash content of the raw material should not exceed 39.5%.

4. The mechanical factory producing parts of the three types uses turning, milling and planing machines. Thus, processing of each part can be carried out by three different technological methods T1, T2 and T3. Time limits for processing the part on the respective machine are given in Table 3.5 for each technological methoda as well as the resources (verst-year) of each group of machine tools. Profit from the sale of each type of product is respectively 22, 18 and 30 UAH. Make the optimal plan for boosting production capacity, which ensures maximum profits provided within the minimum time of lathe use.

Table 3.4

Characteristics of coal, % and indicators of the work of the sites	№ of the site station		
	1	2	3
Ash content	49	37	23
Humidity	7	8	10
Sulfur content	1,8	2,1	3
Costs, gr.	1 184 210	1 381 777	1 083 515
Maximum volume of extraction, ths. tons	1 650	1 090	1 270
Minimal amount of raw material extraction, ths. tons	1 200	600	530

Table 3.5

The type of machine	The rules of time for processing parts, year									Resource time
	I			II			III			
	T1	T2	T3	T1	T2	T3	T1	T2	T3	
Turning	1	0,9	1,1	1,2	1,5	–	0,9	–	–	200
Milling	0,8	0,8	1,3	0,9	1,1	1,3	1,1	0,8	–	400
Planing	–	0,7	0,7	0,7	–	1,3	1,3	0,6	–	300

Task C

Solve the problems of multi-criteria optimization presented in *Task B* by the convolution methods, the main criterion and the successive concession.

SECTION 4

FUZZY SETS AND FUZZY RELATIONS

By the end of this section you will be aware of:

- *concepts of fuzzy set and fuzzy relation;*
- *properties of fuzzy relations;*
- *how to use these concepts in the theory of decision-making.*

4.1 Notion of membership

Let E be a set, where A is its subset. Then $A \subset E$, x is an element of the set E and $x \in A$. To describe this membership, we may use *characteristic function* $\mu_A(x)$, the value of which indicates whether an element x belongs to the set A or not, namely:

$$\mu_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases} \tag{4.1}$$

Example 4.1. Let $E = \{x_1, x_2, x_3, x_4, x_5\}$ and $A = \{x_2, x_3, x_5\}$. We write out for each element of the set E its degree of membership to the set A:

$$\mu_A(x_1) = 0, \quad \mu_A(x_2) = 1, \quad \mu_A(x_3) = 1, \quad \mu_A(x_4) = 0, \quad \mu_A(x_5) = 1.$$

Thus, all the elements of the set A can be described with the elements of the set E, accompanying each of them by the value of its degree of membership, namely:

$$A = \{ (x_1|0), \ (x_2|1), \ (x_3|1), \ (x_4|0), \ (x_5|1) \}.$$

Example 4.2. Let the set $E = [0; 5], \quad A = [1; 2]$, then

$$\mu_A(x) = \begin{cases} 1, & x \in [1; 2], \\ 0, & x \in [0; 1) \cup (2; 5], \end{cases}$$

and the set A can be written like this: $A = \{x \in E : \mu_A(x) = 1\}$.

Let $\overline{A}$ be the complement of the set A with respect to E. Then $\overline{A} \subset E$, $A \cup \overline{A} = E$, and $A \cap \overline{A} = \varnothing$.

If $x \in A$, then $x \notin \overline{A}$, and we can write that when $\mu_A(x) = 1$, then $\mu_{\overline{A}}(x) = 0$.

The following values of the degree of membership of the elements of the set $\overline{A}$ will be obtained for Example 4.1 data:

$$\mu_{\overline{A}}(x_1) = 1,\ \mu_{\overline{A}}(x_2) = 0,\ \mu_{\overline{A}}(x_3) = 0,\ \mu_{\overline{A}}(x_4) = 1,\ \mu_{\overline{A}}(x_5) = 0,$$

and $\overline{A} = \{(x_1|1), (x_2|0), (x_3|0), (x_4|1), (x_5|0)\}$.

For the conditions of Example 4.2

$$\mu_{\overline{A}} = \begin{cases} 1, x \in [0;1) \cup (2;5], \\ 0, x \in [1;2], \end{cases}$$

and $\overline{A} = \{x \in E, \mu_{\overline{A}}(x) = 1\}$.

Now we consider the operations of union and intersection of sets, using the terminology of characteristic functions.

Take set *A* and take set *B*, which characteristic functions are as following:

$$\mu_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A, \end{cases} \qquad \mu_B(x) = \begin{cases} 1, & x \in B, \\ 0, & x \notin B \end{cases}$$

respectively.

The characteristic function of their intersection is function $\mu_{A \cap B}(x)$, which is defined by the following rules:

$$\mu_{A \cap B}(x) = \begin{cases} 1, & x \in A \cap B, \\ 0, & x \notin A \cap B. \end{cases}$$

It can be written in the form of the following formula:

$$\mu_{A \cap B}(x) = \mu_A(x) \cdot \mu_B(x),$$

or

$$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}.$$

Similarly, for union of sets $A \cup B$

$$\mu_{A \cup B}(x) = \begin{cases} 1, & x \in A \cap B, \\ 0, & x \notin A \cap B, \end{cases}$$

that is: $\mu_{A \cup B}(x) = \mu_A(x) \oplus \mu_B(x)$, where $\oplus$ is a Boolean complement, or $\mu_{A\cup B}(x)=\max\{\mu_A(x),\mu_B(x)\}$.

Example 4.3. Consider the following set: $E = \{x_1, x_2, x_3, x_4, x_5\}$, and its two subsets:

$A=\{(x_1|0),(x_2|1),(x_3|1),(x_4|0),(x_5|1)\}$ and $B=\{(x_1|1),(x_2|0),(x_3|1),(x_4|0),(x_5|1)\}$.

Let's find their union and intersection:

$$A\cap B=\{(x_1|0),(x_2|0),(x_3|1),(x_4|0),(x_5|1)\},$$

$$A\cup B=\{(x_1|1),(x_2|1),(x_3|1),(x_4|0),(x_5|1)\},$$

and also, complement the received subsets:

$$\overline{A\cap B}=\{(x_1|1),(x_2|1),(x_3|0),(x_4|1),(x_5|0)\},$$

$$\overline{A\cup B}=\{(x_1|0),(x_2|0),(x_3|1),(x_4|0),(x_5|1)\}.$$

4.2 Definition of fuzzy set and its terminology

In all the examples of the previous subsection, the elements of the set E either belong to or do not belong to the subset A, and the characteristic function acquires the value 0 or 1 of this subset. Now, suppose that it can take any values from the interval [0; 1]. According to this assumption, the element x of the set E may not belong to the set A, then $\mu_A(x)=0$, it can slightly be an element of A [when the value $\mu_A(x)$ is close to 0]; it can belong to a set A of greater or lesser degree [when the value is not very close to 0 and to 1], it can represent an element of the set A to a great extent, while $\mu_A(x)$ close to 1 or finally x can be an element of the set A - and then $\mu_A(x)=1$. Thus, we obtain a generalization of the concept of membership, which allows us to introduce the concept of a fuzzy set.

Definition 4.1. Let E be a set (in the classical conception). A *fuzzy subset A* of E is the set of pairs of the following form: $(x,\mu_A(x))$, where $x\in E$, function $\mu_A(x):E\to[0;1]$. Furthermore, $\mu_A(x)$ is called the *membership function* of the fuzzy subset A.

The value $\mu_A(x)$ of this function for a particular element x is called the *degree of membership* of this element to a fuzzy subset of A.

We designate a fuzzy subset $\widetilde{A}$, or $\widetilde{A} \subset E$, when it is clear that we are talking about fuzzy subsets, we write: $A \subset E$.

The membership of an element to an indistinct subset is affected as follows:

$$x \underset{0,2}{\in} \widetilde{A},\ y \underset{1}{\in} \widetilde{A},\ z \underset{0}{\in} \widetilde{A},$$

where $\underset{0}{\in}$ denotes $\notin$, $\underset{1}{\in}$ is an equivalent $\in$.

Example 4.4. Let $\widetilde{A} = \{(x_1|0{,}2), (x_2|0), (x_3|0{,}3), (x_4|1), (x_5|0{,}5)\}$ is a fuzzy subset of the universal set: $E = \{x_1, x_2, x_3, x_4, x_5\}$.

This means that the fuzzy subset of $\widetilde{A}$ contains elements: x_1, x_3 are insignificant, it does not contain x_2, fully includes the element x_4, and x_5 belongs to it to a large extent.

Thus, we have the opportunity to create a mathematical structure, an object that allows us to operate with relatively incompletely defined elements, the membership of which to a given subset is only hierarchically ordered to a certain degree.

See the examples of similar structures:

– a set of *very high* individuals in a certain set of people;

– a subset of *a dark green* color in a set of all colors;

– a subset of numbers *approximately equal* to a given real number;

– a subset of integers *very close to* 0

– if a is a real number and x is a *small* positive number, then the number $a + x$ forms a fuzzy subset in the set of material numbers.

Note that it is necessary to distinguish between probability and fuzziness.

When it comes to probability, we mean membership or non-membership of an element to a clear, completely defined set under the influence of random conditions. For example, with probability p, a certain student will pass terminal examinations with excellent grades, i.e. s/he will belong to a set of excellent students. A set of excellent students is a definite, clear set. Fuzziness suggests that the set itself is not defined to the full extent. That is why it is impossible to establish its exact limits. An example of such a set is "a set of people who sing *well*”. Here it is unclear the very concept of "*good singing*". In the above examples of fuzzy sets, the elements responsible for their fuzziness are highlighted in *italics*. In fact, one and the same person can be considered "*very tall*" and at the same time not as there is no possibility to clearly define the boundary of this set, and the phrase "*approximately equal*" in each of the situations can be understood in different ways.

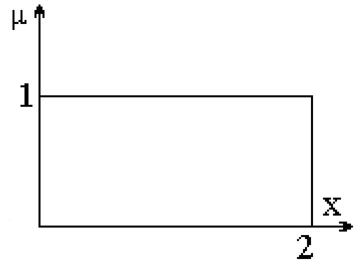
People easily use concepts which cannot be clearly described, and the apparatus of fuzzy sets is designed precisely for the purpose to provide a mathematical form for qualitative concepts to formalize operations with such concepts.

From Definition 4.1, a fuzzy subset is completely described by its membership function. So, below we will sometimes use the membership function to denote a fuzzy set.

Classical sets form a subclass of the class of fuzzy sets. These are the sets which membership functions take values only 0 or 1.

Example 4.5. Consider the classical subset of numbers: $B = \{x \mid 0 \le x \le 2\}$ and a fuzzy subset of numbers: $\tilde{C} = \{x \mid "x \text{ close to } 1"\}$.

The graphs of the membership functions of these sets are shown in Figure 4.1. We would like to note that the form of the membership function μ_C of the fuzzy subset $\tilde{C}$ depends on the meaning, which in this particular situation acquires the concept of "close".



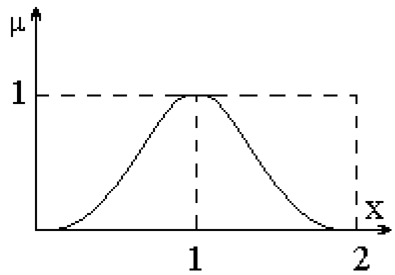


Fig. 4.1. Graphs of membership functions: *a* – the classical set *B*; *b* – fuzzy subset $\tilde{C}$

A fuzzy subset is said to be *empty* if its membership function is zero in the entire set *E*, that is

$$\mu_{\varnothing}(x) = 0, \quad \forall x \in E. \tag{4.1}$$

A universal set *E* can be described by a membership function of the following form:

$$\mu_E(x) = 1, \quad \forall x \in E\ . \tag{4.2}$$

Definition 4.2. *The support* of a fuzzy subset *A* (denoted as supp *A*) with the membership function $\mu_A(x)$ is a set (in the classical sense), which looks like this:

$$\operatorname{supp} A = \{x \mid x \in E,\ \mu_A(x) \ge 0\}. \tag{4.3}$$

Example 4.6. Let universal set $E = \{x_1, x_2, x_3, x_4, x_5\}$, its subset $A = \{(x_1|0{,}1), (x_2|0{,}3), (x_3|0{,}5), (x_4|0), (x_5|1)\}$, then supp $A = \{x_1, x_2, x_3, x_5\}$.

Definition 4.3. A fuzzy subset of A is referred to as *normal* if the following equality is held: $\sup\limits_{x\in E}\mu_A(x)=1$. Otherwise, a fuzzy subset is called *subnormal*.

For example, the fuzzy subset $\widetilde{C}$ of Example 4.2 is normal. Subnormal is often the intersection of fuzzy subsets. Subnormal fuzzy set A can be transformed into normal (to be normalized). To do this, it is necessary to separate the membership function of this set by an amount $\sup\limits_{x\in E}\mu_A(x)$. However, it should be kept in mind that applying such a transformation to any task, it is necessary to clearly imagine its "physical sense".

Definition 4.4. Let $\widetilde{A}$ and $\widetilde{B}$ be fuzzy subsets of the set E, $\mu_A(x)$ and $\mu_B(x)$ be their membership functions, respectively. We say that $\widetilde{A}$ includes $\widetilde{B}$ (that is $\widetilde{B}\subset\widetilde{A}$) if for any element $x\in E$ such an inequality is valid:

$$\mu_B(x)\le\mu_A(x). \tag{4.4}$$

Note: when $\widetilde{B}\subset\widetilde{A}$, then $\operatorname{supp} B\subset\operatorname{supp} A$.

Definition 4.5. The sets A and B coincide (are equivalent) if

$$\mu_B(x)=\mu_A(x),\quad \forall x\in E.$$

Example 4.7. Assume a universal set: $E=\{x_1, x_2, x_3, x_4, x_5\}$. Consider its two subsets:

$$A=\{(x_1|0{,}2),(x_2|0{,}4),(x_3|0),(x_4|1),(x_5|0{,}5)\},$$

$$B=\{(x_1|0),(x_2|0{,}4),(x_3|0),(x_4|0{,}5),(x_5|0{,}2)\}.$$

As $\max\limits_{x\in A}\mu_A(x)=1$, then a fuzzy subset A is a normal subset; for the set B- $\max\limits_{x\in B}\mu_B(x)=0{,}5<1$, that is why the set B is subnormal. Moreover, $B\subset A$, because $\mu_B(x_i)\le\mu_A(x_i),\ \forall x_i\in E$.

Example 4.8. Consider the fuzzy subsets:

A = {x| "x is close to 1"}, B = {x| "x is very close to 1"}.

It is clear that $B\subset A$, then the membership functions of these subsets must satisfy such inequality: $\mu_B(x)\le\mu_A(x),\ \forall x\in E$. Graphically these functions may have a look like it is shown in Figure 4.2.

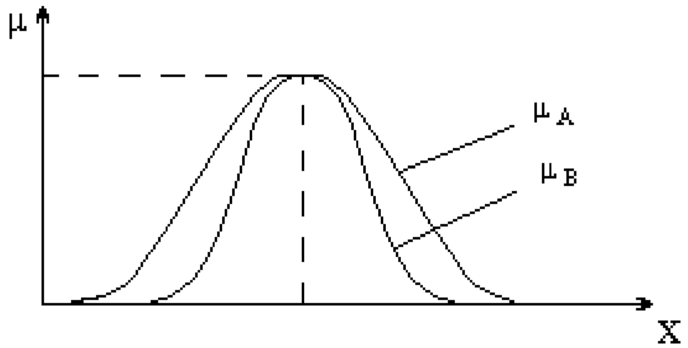


Fig. 4.2. The graphs of membership functions of the sets *A* and *B*, where $B \subset A$

4.3 Operations on fuzzy sets

Since fuzzy sets are an extension of the class of classical sets, then all operations that are defined over classical sets can be applied to them, at the same time there are special operations applicable only for them.

When applied to fuzzy sets, the classical operations, for example, union and intersection, can be defined in various ways. We will look at some of them below. The choice of a particular method depends on the sense that the operation acquires in the framework of the presented task. But, since classical sets represent a subclass of fuzzy, the natural requirement to determine these operations is their correct execution in relation to clear sets.

Definition 4.6. The union of fuzzy subsets *A* and *B* is called fuzzy subsets $A \bigcup B$, the membership function of which has the following form:

$$\mu_{A \bigcup B}(x) = \max\{\mu_A(x), \mu_B(x)\}, \quad x \in E. \tag{4.5}$$

If $\{A_y\}$ is a finite or infinite family of fuzzy subsets with membership functions $\mu_{A_y}(x, y)$, where $y \in Y$ a parameter of the family is, then $C = \bigcap_y A_y$, is a fuzzy set with this membership function:

$$\mu_C(x) = \sup_{y \in Y} \mu_{A_y}(x, y), \; x \in X. \tag{4.6}$$

A graphic interpretation of this definition is shown in Figure 4.3. Here the fuzzy subsets $\widetilde{A}$ and $\widetilde{B}$ are depicted by the graphs of their membership functions, the thick line reflects the membership function of the unification of these sets by Definition 4.6.

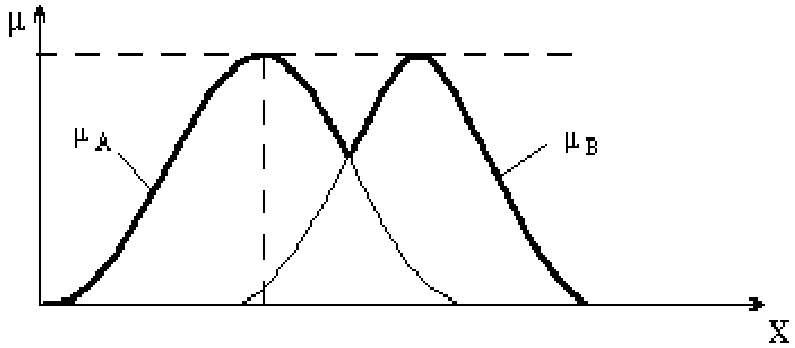


Fig. 4.3. The graph of membership function of union of fuzzy sets *A* and *B*, when $\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(\mathrm{x})\}, \quad x \in E$

Example 4.9. Let fuzzy sets: $A = \{(x_1|1), (x_2|0{,}2), (x_3|0), (x_4|0{,}5), (x_5|0{,}8)\}$ and $B = \{(x_1|0), (x_2|0{,}5), (x_3|0{,}2), (x_4|0{,}2), (x_5|0)\}$ are given for a universal set: $E = \{x_1, x_2, x_3, x_4, x_5\}$. Find their unions.

Solution

According to Definition 4.6

$$A \cup B = \{(x_1|1), (x_2|0{,}5), (x_3|0{,}2), (x_4|0{,}5), (x_5|0{,}8)\}.$$

Definition 4.6, *a*. The union of fuzzy subsets *A* and *B* can also be determined using a limited sum of their membership functions, namely:

$$\mu_{A \cup B}(x) = \begin{cases} 1, & \text{if } \mu_A(x) + \mu_B(x) \geq 1, \\ \mu_A(x) + \mu_B(x) & \text{in other cases.} \end{cases} \tag{4.7}$$

This formula can be written differently as follows:

$$\mu_{A \cup B}(x) = \min\{1, \mu_A(x) + \mu_B(x)\}. \tag{4.8}$$

Graphical interpretation of the union, by definition 4.6, and fuzzy subsets *A* and *B* with membership functions μ_A and μ_B respectively is shown in Figure 4.4.

Example 4.10. Taking the subsets of Example 4.9, we find their union in accordance with Definition 4.6, a. So,

$$A \cup B = \{(x_1|1), (x_2|0{,}7), (x_3|0{,}2), (x_4|0{,}7), (x_5|0{,}8)\}.$$

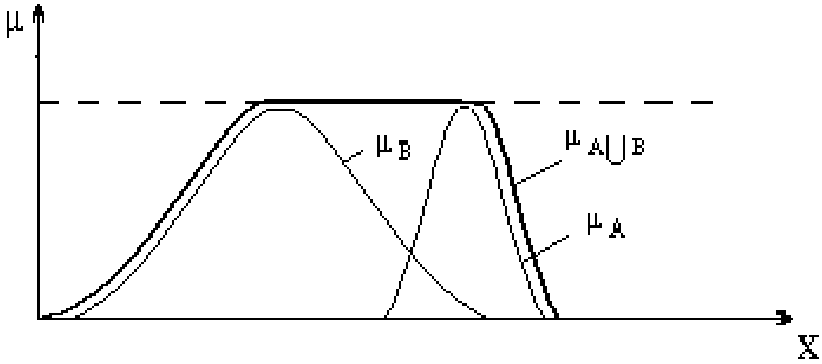


Fig. 4.4. The graph of membership function of the union fuzzy sets *A* and *B* by Definition 4.6, *a*

Definition 4.6, *b*. The union of fuzzy sets can also be found for their algebraic sum that is the merging of fuzzy sets *A* and *B* is a fuzzy set with such membership function:

$$\mu_{A\cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\cdot\mu_B(x). \qquad (4.9)$$

A graphical representation of the membership function of fuzzy subsets *A* and *B* associated in accordance with definition 4.6, *b*, if their membership functions are $\mu_A(x)$ and $\mu_B(x)$ respectively, are shown in Figure 4.5.

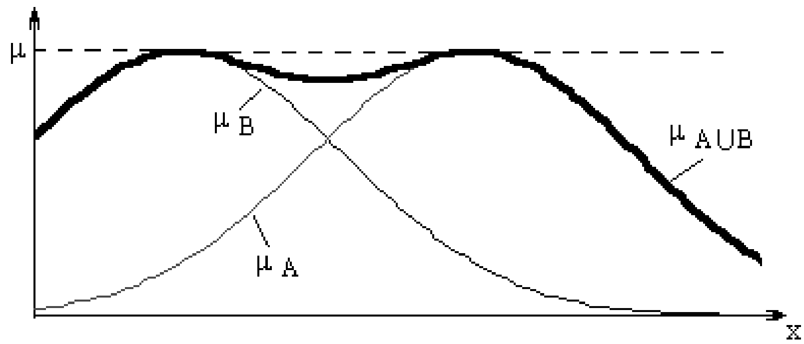


Fig. 4.5. The graph of the membership function of the merged fuzzy sets *A* and *B* by Definition 4.6, *b*

Example 4.11. We find the unions of subsets *A* and *B* from example 4.9 in accordance with Definition 4.6, *b*. Thus,

$$A\cup B = \{(x_1|1),(x_2|0{,}6),(x_3|0{,}2),(x_4|0{,}6),(x_5|0{,}8)\}.$$

Definition 4.7. An intersection of fuzzy subsets *A* and *B* of the universal set *E* is called fuzzy subsets with membership functions of the following form:

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, \quad x \in E. \tag{4.10}$$

Its graph is shown in Figure 4.6

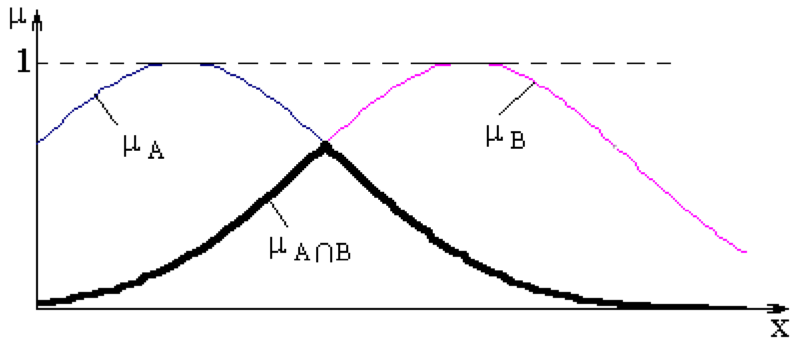


Fig. 4.6. The graph of the intersection membership function of fuzzy subsets *A* and *B* in accordance with Definition 4.7

Example 4.12. Let's define an intersection $A \cap B$ of fuzzy subsets *A* and *B* of the universal set *E*, using Definitions 4.7, if $E = \{x_1, x_2, x_3, x_4, x_5\}$,

$A = \{(x_1|1), (x_2|0{,}4), (x_3|0), (x_4|0{,}7), (x_5|0{,}8)\}$,

$B = \{(x_1|0), (x_2|1), (x_3|0{,}2), (x_4|0{,}4), (x_5|0)\}$.

So, $A \cap B = \{(x_1|0), (x_2|0{,}4), (x_3|0), (x_4|0{,}4), (x_5|0)\}$.

When $\{A_y\}$ is finite or infinite, the family of fuzzy subsets is characterized by membership functions $\mu_{A_y}(x, y)$, where $y \in Y$ is a parameter of the family, then the section $C = \bigcap_y A_y$ of its sets is a fuzzy set which membership function

$$\mu_C(x) = \inf_{y\in Y} \mu_{A_y}(x, y), \quad x \in E. \tag{4.11}$$

The intersection of fuzzy subsets can also be determined in a different way.

Definition 4.7, *a*. The intersection of fuzzy subsets *A* and *B* is a limited product of their membership functions, that is,

$$\mu_{A\cap B}(x) = \max\{0\,, \mu_A(x) + \mu_B(x) - 1\}. \tag{4.12}$$

A graphic interpretation of this intersection is shown in Figure 4.7. Here, the functions of membership to fuzzy subsets *A* and *B*, respectively, are reflected. The thick line shows the function of the intersection of *A* and *B*.

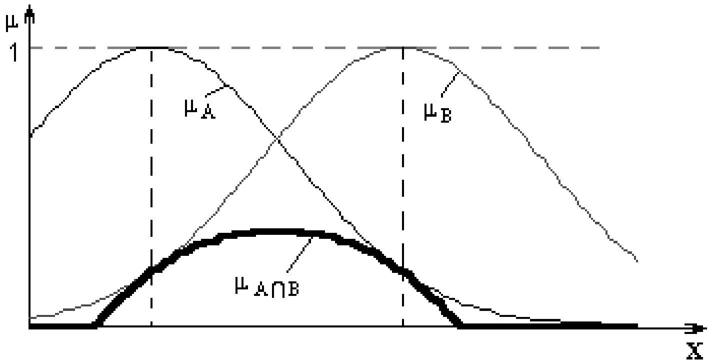


Fig. 4.7. The graph of the intersection function of fuzzy sets by Definition 4.7, *a*

Another definition of the intersection can be formulated using the algebraic product of their membership functions.

Definition 4.7, *b*. An intersection of fuzzy sets *A* and *B* is referred to as a fuzzy set which membership function is equal to the algebraic product of the membership functions of these sets, that is

$$\mu_{A \cap B}(x) = \mu_{\mathrm{A}}(x)\mu_B(x). \qquad (4.13)$$

Graphs of membership functions $\mu_A(x)$ and $\mu_B(x)$ fuzzy sets *A* and *B*, and their intersections by definition 4.7, *b* is shown in Figure 4.8. The thick line of the graph corresponds to the section membership function.

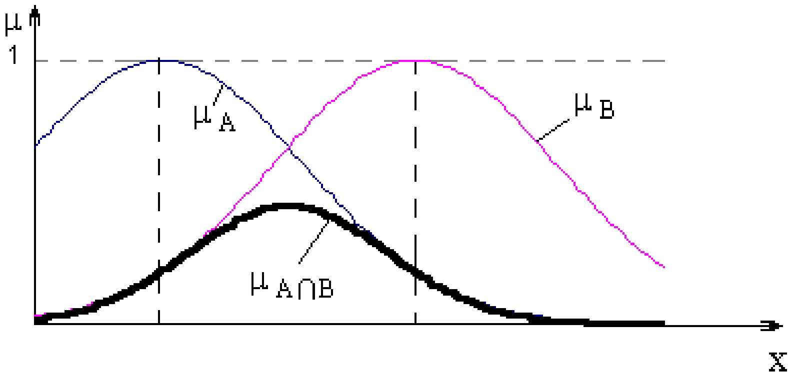


Fig. 4.8. The graph of intersection membership function of fuzzy sets *A* and *B* by Definition 4.7, *b*

Example 4.13. Let`s find the intersection of fuzzy subsets *A* and *B* if $A \subset E$, $B \subset E$, $E = \{x_1, x_2, x_3, x_4, x_5\}$, $A = \{(x_1|1), (x_2|0{,}4), (x_3|0), (x_4|0{,}7), (x_5|0{,}8)\}$, $B = \{(x_1|0), (x_2|1), (x_3|0{,}2), (x_4|0{,}4), (x_5|0)\}$.

Then, by Definitions 4.7

$$A \cap B = \{(x_1|0), (x_2|0{,}4), (x_3|0), (x_4|0{,}1), (x_5|0)\},$$

and by Definitions 4.7, *b*

$$A \cap B = \{(x_1|0), (x_2|0{,}4), (x_3|0), (x_4|0{,}28), (x_5|0)\}.$$

Definition 4.8. A *complement* of a fuzzy set A in E is referred to as a fuzzy set $\overline{A}$, characterized by such a function:

$$\mu_A(x) = 1 - \mu_A(x),\ x \in E. \tag{4.14}$$

Note that the property: $A \cap \overline{A} = \varnothing$ hold for classical sets under all conditions is not always valid with respect to fuzzy sets. For example, if the compliment of a fuzzy set is defined as described above and the intersection is calculated according to Rule 4.7 or 4.7, b, then $A \cap \overline{A} \neq \varnothing$, but in calculating the intersection according to Rule 4.7, and the equality $A \cap \overline{A} = \varnothing$ will be correct. Options of intersection and merging are chosen by a researcher, depending on what properties of operations are essential for the problem being solved.

Example 4.18. Consider such a fuzzy subset: A = {a number much larger than 0}, its membership function is shown in Figure 4.9 with a solid curve. The complement of the set A is a fuzzy set of numbers, not much greater than zero. This set corresponds to the membership function, the graph of which is shown in Figure 4.9 with a dotted line.

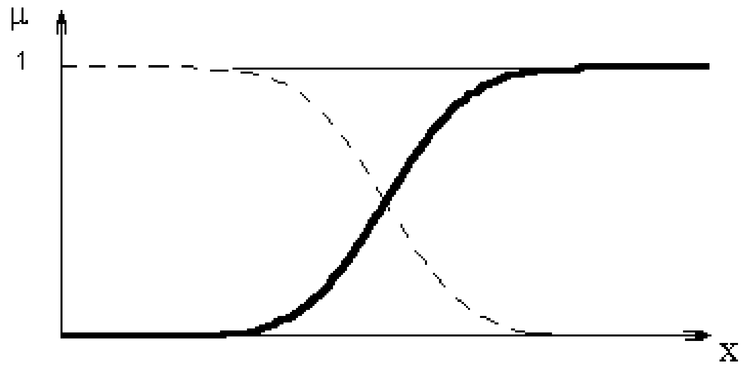


Fig. 4.9. The graphs of the membership functions of the fuzzy set A and its complement (for example 4.18)

The non-empty intersection of the sets A and $\overline{A}$ in this example is a fuzzy set of numbers "*significantly greater than zero and not much greater than zero*" simultaneously. The non-emptiness of this fuzzy set reflects the fact that the very concept of "*being significantly larger*" is not clearly described. So, some numbers may belong to both sets at the same time. In some sense, this intersection can be considered a "fuzzy boundary" between the sets A and $\overline{A}$.

Definition 4.9. The *difference* of subsets A and B of the universal set E is a fuzzy set $A \setminus B$, characterized by such a membership function:

$$\mu_{A\setminus B}(x) = \begin{cases} \mu_A(x) - \mu_B(x), & \text{if} \quad \mu_A(x) \geq \mu_B(x), \\ 0 & \text{in other case,} \end{cases} \tag{4.15}$$

that is

$$\mu_{A\setminus B}(x) = \max\{\mu_{\mathrm{A}}(x) - \mu_B(x), 0\}. \tag{4.16}$$

Let's find the difference between fuzzy subsets A and B, if $A \subset E$, $B \subset E$, $E = \{x_1, x_2, x_3, x_4, x_5\}$,

$A = \{(x_1|1), (x_2|0{,}4), (x_3|0), (x_4|0{,}7), (x_5|0{,}8)\}$,

$B = \{(x_1|0), (x_2|1), (x_3|0{,}2), (x_4|0{,}4), (x_5|0)\}$,

Then, by Definitions 4.9, a

$A \setminus B = \{(x_1 \,|\, 1), (x_2 \,|\, 0), (x_3 \,|\, 0), (x_4 \,|\, 0{,}3), (x_5 \,|\, 0{,}8)\}$.

4.4 Distance between fuzzy subsets

Hamming distance. First, we recall the notion of Hamming distance in its application to classical subsets.

Let A and B be two classical subsets of a finite set: $E = \{x_1, \ldots, x_7\}$, and

$B = \{(x_1|0), (x_2|1), (x_3|0), (x_4|0), (x_5|0), (x_6|1), (x_7|1)\}$,

$A = \{(x_1|1), (x_2|0), (x_3|0), (x_4|1), (x_5|0), (x_6|1), (x_7|0)\}$.

Under the *Hamming distance* between A and B, we mean this value:

$$d(A,B) = \sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)|. \tag{4.17}$$

In our example

$$d(A,B) = |1-0| + |0-1| + |0-0| + |1-0| + |0-0| + |1-1| + |0-1| = 4.$$

The Hamming distance satisfies all the axioms of the metric, namely:

1. $d(X,Y) \geq 0$,

2. $d(X,Y) = d(Y,X)$,

3. $d(X,Z) \le d(X,Y) + d(Y,Z)$,

4. $d(X,X) = 0$.

Task. Check the fulfillment of these axioms with respect to Hamming distance.

For a finite set E which cardinality is $m(E) = n$ (n is the number of elements of the set E), we also define the relative Hamming distance as follows:

$$\delta(A,B) = \left(\frac{1}{n}\right) d(A,B). \tag{4.18}$$

In respect to the subsets A and B above $\delta(A,B) = \frac{1}{7} \cdot d(A,B) = \frac{4}{7}$.

It is obvious that always $0 \le \delta(A, B) \le 1$.

Generalization of the concept of Hamming distance

Consider now three fuzzy subsets $A, B, C \subset E$, here E is a finite set of cardinality n, namely:

$$A = \frac{x_1 \mid x_2 \mid x_3 \mid \ldots x_n}{a_1 \mid a_2 \mid a_3 \mid \ldots a_n}, \tag{4.19}$$

$$B = \frac{x_1 \mid x_2 \mid x_3 \mid \ldots x_n}{b_1 \mid b_2 \mid b_3 \mid \ldots b_n}, \tag{4.20}$$

$$C = \frac{x_1 \mid x_2 \mid x_3 \mid \ldots x_n}{c_1 \mid c_2 \mid c_3 \mid \ldots c_n}. \tag{4.21}$$

Suppose that we have determined the distance D (a_i, b_i) between a_i and b_i, $i = \overline{1,n}$ and also between (b_i, c_i) and (*ai, ci*), $i = \overline{1,n}$. These distances will correspond to such irregularities:

$$D(a_i, c_i) \le D(a_i, b_i) + D(b_i, c_i),\ \forall\, i = 1, 2, \ldots, n. \tag{4.22}$$

Moreover, we can write down that

$$\sum_{i=1}^{n} D(a_i, c_i) \le \sum_{i=1}^{n} D(a_i, b_i) + \sum_{i=1}^{n} D(b_i, c_i), \tag{4.23}$$

and

$$\sum_{i=1}^{n} D(a_i, c_i) \le \sum_{i=1}^{n} D(a_i, b_i) + \sum_{i=1}^{n} D(b_i, c_i), \qquad (4.23)$$

These two formulae give two estimates of the distance between subsets, expression (4.23) gives a linear estimate, and (4.24) is quadratic.

Let’s consider the case when the membership functions of fuzzy subsets acquire their values in the interval [0; 1], that is, when in the expressions (4.19) - (4.21) the quantities a_i, b_i, $c_i \in [0; 1]$, $i = 1, 2, \dots n$.

Let's pretend that $D(a_i, b_i) = |a_i - b_i|$, $D(b_i, c_i) = |b_i - c_i|$, $D(a_i, c_i) = |a_i - c_i|$

We define two types of distances.

Definition 4.10. The generalized Hamming distance or the *linear distance* is calculated by the formula:

$$d(A, B) = \sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)|. \qquad (4.25)$$

It's obvious that

$$0 \le d(A, B) \le n. \qquad (4.26)$$

Definition 4.11. The *Euclidean* or *quadratic* distance is calculated as follows:

$$e(A, B) = \sqrt{\sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2}. \qquad (4.27)$$

This distance satisfies this condition:

$$0 \le e(A, B) \le \sqrt{n}. \qquad (4.28)$$

We also determine the relative distances

The generalized relative Hamming distance

$$\delta(A, B) = \frac{d(A, B)}{n} = \frac{1}{n} \sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)|, \qquad (4.29)$$

for this will the following inequality be valid: $0 \le \delta(A, B) \le 1$.

The relative Euclidean distance

$$\varepsilon(A, B) = \frac{e(A, B)}{\sqrt{n}} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2}, \qquad (4.30)$$

$$0 \le \varepsilon(A, B) \le 1.$$

The choice of a particular distance depends on the nature of the problem discussed. Each of these distances has advantages and disadvantages, which become clear in their applications. Obviously, you can specify other distances too.

Example 4.19. Determine the distance between such fuzzy sets:

$$A = \frac{x_1}{0{,}7} \left| \frac{x_2}{0{,}2} \right| \frac{x_3}{0} \left| \frac{x_4}{0{,}6} \right| \frac{x_5}{0{,}5} \left| \frac{x_6}{1} \right| \frac{x_7}{0}, \qquad B = \frac{x_1}{0{,}2} \left| \frac{x_2}{0} \right| \frac{x_3}{0} \left| \frac{x_4}{0{,}6} \right| \frac{x_5}{0{,}8} \left| \frac{x_6}{0{,}4} \right| \frac{x_7}{1}.$$

Solving

$d(A, B) = |0{,}7 - 0{,}2| + |0{,}2 - 0| + |0 - 0| + |0{,}6 - 0{,}6| + |0{,}5 - 0{,}8| + |1 - 0{,}4| +$

$+ |0 - 1| = 0{,}5 + 0{,}2 + 0{,}3 + 0{,}6 + 1 = 2{,}6.$

$$\delta(A,B) = \frac{1}{7} d(A,B) = \frac{2{,}6}{7} = 0{,}37.$$

$$e(A,B) = \sqrt{(0{,}7-0{,}2)^2 + (0{,}2-0)^2 + (0-0)^2 + (0{,}6-0{,}6)^2 + (0{,}5-0{,}8)^2 + (1-0{,}4)^2 + (0-1)^2} =$$

$$= \sqrt{(0{,}5)^2 + 0{,}2^2 + 0{,}3^2 + 0{,}6^2 + 1} = \sqrt{0{,}25 + 0{,}04 + 0{,}09 + 0{,}36 + 1} = \sqrt{1{,}74} = 1{,}32,$$

$e(A, B) = 1{,}32,$

$$\varepsilon\left(A,B\right) = \frac{1}{\sqrt{7}} \cdot 1{,}32 = 0{,}49.$$

The distances $d(A, B)$, $e(A, B)$ can be defined also when the universal set is infinite (countable or not) if the corresponding sums and integrals coincide. If E is countable, then

$$d(A,B) = \sum_{i=1}^{\infty} \left|\mu_A(x_i) - \mu_B(x_i)\right|, \tag{4.31}$$

$$e(A,B) = \sqrt{\sum_{i=1}^{\infty} (\mu_A(x_i) - \mu_B(x_i))^2}, \tag{4.32}$$

when these series coincide.

If $E = R$, then

$$d(A,B) = \int_{-\infty}^{+\infty} \left|\mu_A(x) - \mu_B(x)\right| dx \tag{4.33}$$

and

$$e(A,B) = \sqrt{\int_{-\infty}^{+\infty} (\mu_A(x) - \mu_B(x))^2 \, dx}, \tag{4.34}$$

when the integrals coincide.

If $E \subset R$ is bounded above and below, then the corresponding integrals always coincide, and the distances $d\,(A, B)$ and $e\,(A, B)$ are finite. Then you can determine the relative distances, namely:

$$\delta(A,B) = \frac{d(A,B)}{\beta - \alpha}, \tag{4.35}$$

$$\varepsilon(A,B) = \frac{e(A,B)}{\beta - \alpha}, \tag{4.36}$$

where $d(A,B) = \int_{\alpha}^{\beta} |\mu_A(x) - \mu_B(x)| dx$ and $e(A,B) = \sqrt{\int_{\alpha}^{\beta} (\mu_A(x) - \mu_B(x))^2 dx}$.

Consider as an example a geometric interpretation of the concept of distance between fuzzy sets. Let the sets $\widetilde{A}$ and $\widetilde{B}$ are subsets of the universal set $E \subset R^1$, $E = [\alpha, \beta]$, and the graphs of their membership functions $\mu_A(x)$ and $\mu_B(x)$ are shown in Figure 4.10. Then the linear distance between these sets corresponds to the area of the shaded figure bounded by the curves of the functions $\mu_A(x)$ and $\mu_B(x)$.

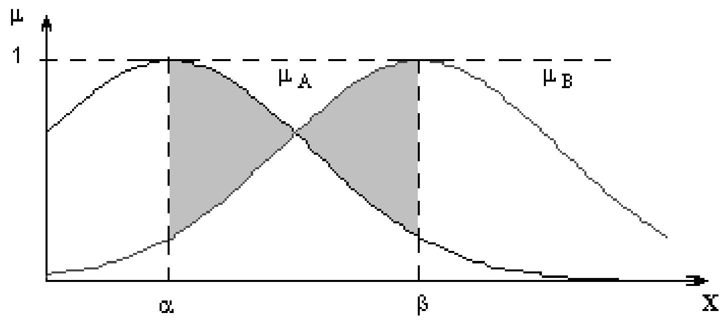


Fig. 4.10. Geometric interpretation of the linear distance between fuzzy sets

4.5 Classical subset closest to fuzzy. Fuzzy index

The question arises as to which classical subset (or subsets) is at the Euclidean distance from a given fuzzy set A. It is easy to see that this is an classical subset (it is denoted $\underline{A}$) for which

$$\mu_{\underline{A}}(x_i) = \begin{cases} 0\ , \ \text{if}\ \mu_A(x_i) < 0{,}5\ , \\ 1\ , \ \text{if}\ \ \mu_A(x_i) > 0{,}5\ , \\ 0\ \text{ or } 1\ , \ \text{if}\ \ \mu_A(x_i) = 0{,}5. \end{cases} \tag{4.39}$$

In order to avoid inaccuracy, we specify that $\mu_{\underline{A}}(x_i) = 0$, when $\mu_A(x_i) = 0{,}5$ Thus, we can formulate a definition.

Definition 4.12. The set $\underline{A}$, which is closest to fuzzy set *A*, is called the set $\underline{A}$ characterized by the membership function:

$$\mu_{\underline{A}}(x_i) = \begin{cases} 0\,, & \text{if } \mu_A(x_i) \le 0{,}5; \\ 1\,, & \text{if } \mu_A(x_i) > 0{,}5. \end{cases} \tag{4.40}$$

Example 4.20. Let $E = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, $A \subset E$ and

$A = \{(x_1|0{,}2), (x_2|0{,}8), (x_3|0{,}5), (x_4|0{,}3), (x_5|1), (x_6|0), (x_7|0{,}9), (x_8|0{,}4)\}$.

The classical set closest to the set *A* will be:

$\underline{A} = \{(x_1|0), (x_2|1), (x_3|0), (x_4|0), (x_5|1), (x_6|0), (x_7|1), (x_8|0)\}$.

Using the notions of distances introduced earlier for fuzzy subsets, we define two *indices of fuzziness*.

The linear index of fuzziness is determined through the generalized relative Hamming distance as follows:

$$\nu(A) = \frac{2}{n} d(A, \underline{A}). \tag{4.41}$$

The quadratic fuzzy index is determined via the relative Euclidean distance, namely:

$$\eta(A) = \frac{2}{\sqrt{n}} e(A, \underline{A}). \tag{4.42}$$

Factor 2 in the numerator is introduced in order to ensure the content of the fuzzy index within the following limits:

$$0 \le \nu(A, \underline{A}) \le 1, \tag{4.43}$$

$$0 \le \eta(A, \underline{A}) \le 1. \tag{4.44}$$

When $E = [a, b] \subset R$ then the linear index of fuzziness is calculated by the formula:

$$\nu(A, B) = \frac{2}{b-a} \int_a^b |\mu_A(x) - \mu_{\underline{A}}(x)| dx. \tag{4.45}$$

The geometric interpretation of the nearest classical set and the fuzzy index can be seen in Figure 4.11. Here, the thick line shows the membership function of the nearest classical set $\underline{A}$ to the fuzzy set A described by the membership function μ_A. The linear fuzzy index corresponds to the normalized area of the shaded figure.

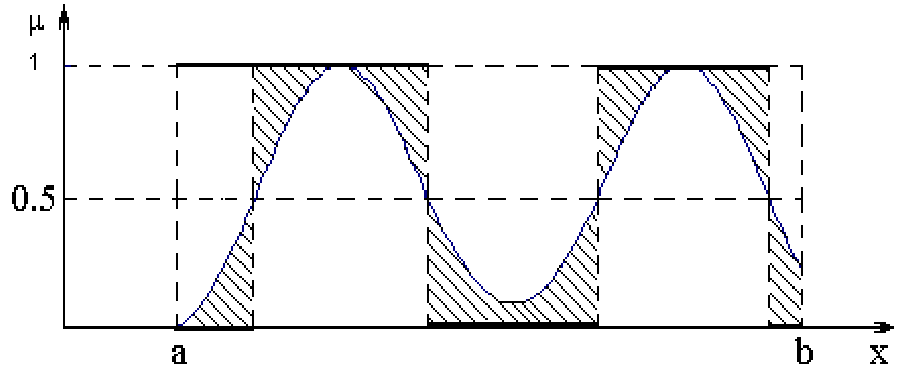


Fig. 4.11. Geometric interpretation of the index of fuzziness

Indices of fuzziness can be determined in another way, namely:

$$\nu(A) = \frac{2}{n}\sum_{i=1}^{n} \min(\mu_A(x_i), \mu_{\overline{A}}(x_i)), \tag{4.46}$$

$$\eta(A) = \frac{2}{\sqrt{n}}\sqrt{\sum_{i=1}^{n} \min\left\{\mu_A^2(x_i), \mu_{\overline{A}}^2(x_i)\right\}}. \tag{4.47}$$

Indeed, for any element $x_i \in E$

$$\left| \mu_A(x_i) - \mu_{\underline{A}}(x_i) \right| = \mu_{A \cap \overline{A}}(x_i). \tag{4.48}$$

Then, the formula (4.41) for calculating the linear index of fuzziness can be rewritten in a convenient form, namely:

$$\nu(A) = \frac{2}{n}\sum_{i=1}^{n} \mu_{A \cap \overline{A}}(x_i). \tag{4.49}$$

From this expression 4.49 it becomes obvious that $\nu(A) = \nu(\overline{A})$.

Example 4.21. Let $E = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, $A \subset E$,

$A = \{(x_1|0{,}2), (x_2|0{,}8), (x_3|0{,}5), (x_4|0{,}3), (x_5|1), (x_6|0), (x_7|0{,}9), (x_8|0{,}4)\}$,

$\overline{A} = \{(x_1|0{,}8), (x_2|0{,}2), (x_3|0{,}5), (x_4|0{,}7), (x_5|0), (x_6|1), (x_7|0{,}1), (x_8|0{,}6)\}$, Let’s calculate the fuzzy index of the set A. We will first determine the intersection of the sets presented above.

$$A \cap \overline{A} = \{(x_1|0,2),(x_2|0,2),(x_3|0,5),(x_4|0,3),(x_5|0),(x_6|0),(x_7|0,1),(x_8|0,4)\}.$$

Now we can calculate the linear index of fuzziness:

$$\nu(A) = \frac{2}{8}(0,2+0,2+0,5+0,3+0+0+0,1+0,4) = 0,425.$$

Let A and B are two fuzzy subsets of E. Let us explain how the indices of the fuzziness of the intersection $A \cap B$ and the union $A \cup B$ of these fuzzy subsets correspond to the indecision indices of the original subsets?

Let's consider examples.

Example 4.22. Let $E = \{x_1, x_2, x_3\}$, $A = \{(x_1|0,2),(x_2|0,6),(x_3|0,1)\}$, $B = \{(x_1|0,6),(x_2|0,3),(x_3|0,8)\}$. We calculate the indices of the fuzziness of the original sets and their intersections, namely:

$$\overline{A} = \{(x_1|0,8),(x_2|0,4),(x_3|0,9)\}, \quad \nu(A) = \frac{2}{3}(0,2+0,4+0,1) \approx 0,46.$$

$$\overline{B} = \{(x_1|0,4),(x_2|0,7),(x_3|0,2)\}, \quad \nu(B) = \frac{2}{3}(0,4+0,3+0,2) \approx 0,6.$$

$$A \cap B = \{(x_1|0,2),(x_2|0,3),(x_3|0,1)\}, \ \overline{A \cap B} = \{(x_1|0,8),(x_2|0,7),(x_3|0,9)\},$$

$$\nu(A \cap B) = \frac{2}{3}(0,2+0,3+0,1) \approx 0,4.$$

Obviously, in this case, the fuzziness index of the intersection is less than the index of fuzziness of the original subsets.

Example 4.23 Let $E = \{x_1, x_2, x_3\}$, $A' = \{(x_1|0,8),(x_2|0,6),(x_3|0,8)\}$, $B' = \{(x_1|0,4),(x_2|0,7),(x_3|0,2)\}$. We calculate the indices of fuzziness of these sets and their intersections, i.e.:

$$\overline{A'} = \{(x_1|0,2),(x_2|0,4),(x_3|0,2)\}, \qquad \nu(A') = \frac{2}{3}(0,2+0,4+0,2) \approx 0,53,$$

$$\overline{B'} = \{(x_1|0,6),(x_2|0,3),(x_3|0,8)\}, \qquad \nu(B') = \frac{2}{3}(0,4+0,3+0,2) = 0,60,$$

$$A' \cap B' = \{(x_1|0,4),(x_2|0,6),(x_3|0,2)\}, \quad \overline{A' \cap B'} = \{(x_1|0,6),(x_2|0,4),(x_3|0,8)\}$$

$$\nu(A' \cap B') = \frac{2}{3}(0{,}4 + 0{,}4 + 0{,}2) \approx 0{,}66 .$$

In this example, the fuzzy intersection index is greater than the fuzzy indexes of the original subsets.

Thus, we see that the fuzziness index of the intersection of subsets A and B can be either smaller or larger than the indices of fuzziness of the original subsets. The same can be said about union fuzzy subsets. The statement is also fair for the quadratic fuzzy index.

4.6 Classical subset of α-level of fuzzy set

Definition 4.13. Let $\alpha \in [0; 1]$. The classical set of α-level of fuzzy subset A (denoted by) is the set consisting only of those elements of A whose membership function is not less than α, that is,

$$A_\alpha = \{x \,|\, \mu_A(x) \geq \alpha\} . \tag{4.50}$$

Example 4.24. Let the fuzzy set A be given in the following form:

$$A = \frac{x_1 \,|\, x_2 \,|\, x_3 \,|\, x_4 \,|\, x_5 \,|\, x_6 \,|\, x_7}{0{,}8 \,|\, 0{,}1 \,|\, 1 \,|\, 0{,}3 \,|\, 0{,}6 \,|\, 0{,}2 \,|\, 0{,}5} .$$

We define sets of level 0,3 and 0,5 of this fuzzy subset, namely:

$$A_{0,3} = \frac{x_1 \,|\, x_2 \,|\, x_3 \,|\, x_4 \,|\, x_5 \,|\, x_6 \,|\, x_7}{1 \,|\, 0 \,|\, 1 \,|\, 1 \,|\, 1 \,|\, 0 \,|\, 1} , \qquad A_{0,3} = \{x_1, x_3, x_4, x_5, x_7\},$$

$$A_{0,5} = \frac{x_1 \,|\, x_2 \,|\, x_3 \,|\, x_4 \,|\, x_5 \,|\, x_6 \,|\, x_7}{1 \,|\, 0 \,|\, 1 \,|\, 0 \,|\, 1 \,|\, 0 \,|\, 1} , \qquad A_{0,5} = \{x_1, x_3, x_5, x_7\}.$$

Example 4.25. Let the universal set $X = \{1, 2, ..., 6\}$, and the membership function of the fuzzy set $A \subset X$ is given by the table:

x	0	1	2	3	4	5	6
$\mu_A(x)$	0	0,1	0,3	0,5	0,7	0,9	1

Then, for the set A we can write down such level sets:

$A_{0,1} = \{1, 2, 3, 4, 5, 6\}$, $A_{0,3} = \{2, 3, 4, 5, 6\}$, $A_{0,5} = \{3, 4, 5, 6\}$,

$A_{0,7} = \{4, 5, 6\}$, $A_{0,9} = \{5, 6\}$, $A_1 = \{6\}$.

Example 4.26. Let $X = R^+$ is the graph of the membership function μ_A of the fuzzy set A is shown in Figure 4.12, *a*. The sets of level α_1 and α_2 and the graphs of their membership functions are shown in Figure 4.12, *b* and 4.12, *c*.

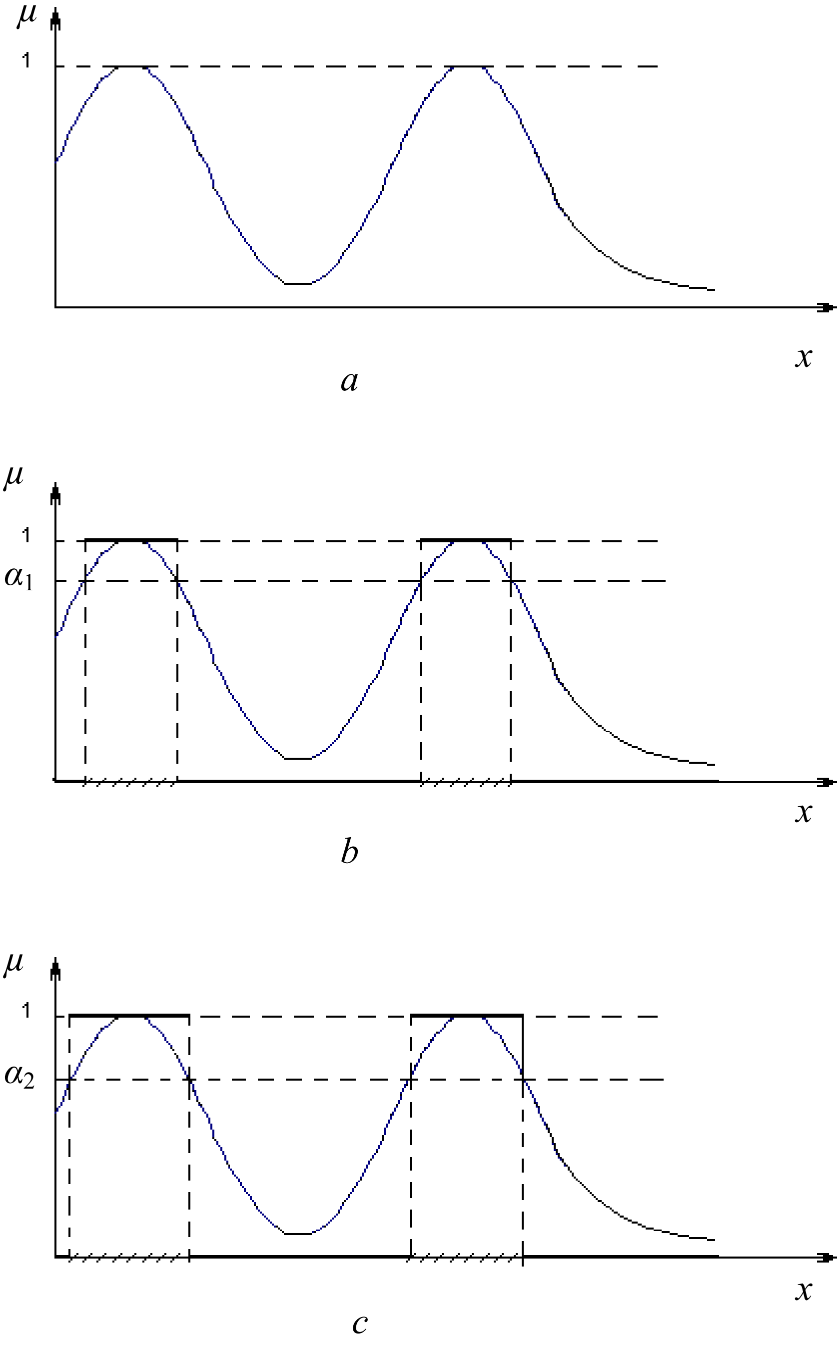


Fig. 4.12. The level sets and their relation

As can be seen from these examples, for all values of α_1 and α_2 satisfying the following conditions: $0 < \alpha_1 \le 1$, $0 < \alpha_2 \le 1$ and $\alpha_2 < \alpha_1$, the corresponding level sets A_{α_1} and A_{α_2} are related by the following relation: $A_{\alpha_1} \subset A_{\alpha_2}$.

It is convenient to use level sets for the formulation and analysis of some decision-making problems, we will also apply them in solving problems of fuzzy mathematical programming.

Let $(A \bigcup B)_\alpha$ and $(A \bigcap B)_\alpha$ are sets of α-level of union and intersection of fuzzy sets A and B respectively. Let us consider their connection with level sets A_α and B_α output sets. If for the operations of intersection and union we take definition 4.7 and 4.6 respectively, then this relationship will be:

$$(A \bigcup B)_\alpha = A_\alpha \bigcup B_\alpha, \quad (A \bigcap B)_\alpha = A_\alpha \bigcap B_\alpha.$$

When using definitions 4.6, *b* and 4.7, only inclusion will be fair, i.e.:

$$(A \bigcup B)_\alpha \supset A_\alpha \bigcup B_\alpha, \quad (A \bigcap B)_\alpha \subset A_\alpha \bigcap B_\alpha.$$

For fuzzy subsets, the decomposition theorem will be carried out below.

Theorem 4.1. Any fuzzy subset A can be decomposed into level sets, that is, submit it in this form:

$$A = \bigcup_\alpha \alpha A_\alpha, \tag{4.51}$$

moreover, the membership function of the set αA_α: $\mu_{\alpha A_\alpha}(x) = \alpha \mu_{A_\alpha}(x)$, the union of fuzzy sets is satisfied for all values $\alpha \in [0; 1]$, and the membership function of the level set α is given as follows:

$$\mu_{A_\alpha}(x) = \begin{cases} 1, \mu_A(x) \geq \alpha, \\ 0, \mu_A(x) < \alpha. \end{cases}$$

Example 4.27. For the set A and its level sets, using example 4.25, we can write that

$A = 0{,}1\{1, 2, 3, 4, 5, 6\} \bigcup 0{,}3\{2, 3, 4, 5, 6\} \bigcup 0{,}5\{3, 4, 5, 6\} \bigcup 0{,}7\{4, 5, 6\} \bigcup$

$\bigcup 0{,}8\{5, 6\} \bigcup 1\{6\}$.

The decomposition formula will also be correct when the universal set has the cardinality of the continuum.

Example 4.28. Let the fuzzy set $A \subset R^+$ is given by its membership function: $\mu_A(x) = 1 - \dfrac{1}{1 + x^2}$, $x \in R^+$. Having considered the interval $[\alpha; 1]$, where $0 < \alpha \leq 1$, we can make the following conclusion:

$$\mu_{A_\alpha}(x_i)=\begin{cases}1\ ,\ \text{if}\ \ \mu_A(x_i)\in[\alpha;1],\\ 1\ ,\ \text{if}\ \ \mu_A(x_i)\notin[\alpha;1].\end{cases}$$

So, in this example

$$\mu_{A_\alpha}(x)=\begin{cases}1,\ \text{if}\ x\geq\sqrt{\dfrac{\alpha}{1-\alpha}},\\ 0,\ \text{if}\ x<\sqrt{\dfrac{\alpha}{1-\alpha}}\ .\end{cases}$$

The decomposition theorem is applicable not only to analysis, but also to the synthesis of fuzzy sets.

Consider the sequence of classical subsets $A_1\subset A_2\subset A_3 \ldots \subset A_n$, and set the value of α_1 for the set A_1, α_2 for the set A_2, and α_n for A_n, and $\alpha_1 > \alpha_2 > \ldots > \alpha_n$, then, using the formula (4.51), we obtain a fuzzy subset of A.

Example 4.29. Suppose it is given and classical set:

$$X=\{x_1, x_2, x_3, \ldots, x_{10}\},$$

and their subsets:

$A_1 = \{x_1, x_4, x_5, x_7, x_9\}$,

$A_2 = \{x_1, x_4, x_5, x_6, x_7, x_9\}$,

$A_3 = \{x_1, x_2, x_4, x_5, x_6, x_7, x_9\}$,

$A_4 = \{x_1, x_2, x_4, x_5, x_6, x_7, x_9, x_{10}\}$,

and the following the numbers are also defined $\alpha_1 = 0{,}9$, $\alpha_2 = 0{,}5$, $\alpha_3 = 0{,}4$, $\alpha_4 = 0{,}1$.

Using the formula (4.51), we obtain a fuzzy set A.

Let's construct the sets $\alpha_i A_i$ first by the following formula:

$$\mu_{\alpha_i A_i}(x_j)=\alpha_i\mu_{A_i}(x_j)=\begin{cases}\alpha_i,\ \text{if}\, x_j\in A_i\ ,\\ 0,\ \ \text{if}\, x_j\ \overline{\in}\ A_i.\end{cases}$$

Then we obtain the following subsets:

$\alpha_1 A_1 =$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
0,9	0	0	0,9	0,9	0	0,9	0	0,9	0

,

$\alpha_2 A_2 =$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
0,5	0	0	0,5	0,5	0,5	0,5	0	0,5	0

,

$$\alpha_3 A_3 = \frac{\begin{array}{c|c|c|c|c|c|c|c|c|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{array}}{\begin{array}{c|c|c|c|c|c|c|c|c|c} 0{,}4 & 0{,}4 & 0 & 0{,}4 & 0{,}4 & 0{,}4 & 0{,}4 & 0 & 0{,}4 & 0 \end{array}},$$

$$\alpha_4 A_4 = \frac{\begin{array}{c|c|c|c|c|c|c|c|c|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{array}}{\begin{array}{c|c|c|c|c|c|c|c|c|c} 0{,}1 & 0{,}1 & 0 & 0{,}1 & 0{,}1 & 0{,}1 & 0{,}1 & 0 & 0{,}1 & 0{,}1 \end{array}}.$$

Combining these fuzzy sets, we obtain the required fuzzy set, that is

$$A = \bigcup_{\alpha_i} \alpha_i A_i \,,$$

$$A = \frac{\begin{array}{c|c|c|c|c|c|c|c|c|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{array}}{\begin{array}{c|c|c|c|c|c|c|c|c|c} 0{,}9 & 0{,}4 & 0 & 0{,}9 & 0{,}9 & 0{,}5 & 0{,}9 & 0 & 0{,}9 & 0{,}1 \end{array}}.$$

4.7 Special operations on fuzzy sets

We have already considered a number of operations on fuzzy sets. They were similar to operations with classical sets, but acting as a new structure, fuzzy sets have new properties. Therefore, new operations can be introduced on them that do not make sense for classical sets.

We first define the Cartesian product of fuzzy sets.

Definition 4.14. The *Cartesian product* of $A_1 \times A_2 \times ... \times A_n$ fuzzy sets $A_i \subset X_i$, i = 1, ..., n is a fuzzy set A in the Cartesian product $X_1 \times X_2 \times \ldots \times X_n$, the membership function of which has the following form:

$$\mu_A(x) = \min\{\mu_{A_1}(x_1),..\mu_{An}(x_n)\}, \; x = (x_1, x_2, \ldots, x_n) \in X. \tag{4.52}$$

Example 4.30. We define the Cartesian product of fuzzy sets A and B if

$$A = \frac{\begin{array}{c|c|c|c|c} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}}{\begin{array}{c|c|c|c|c} 0{,}1 & 0{,}5 & 0{,}3 & 1 & 0{,}2 \end{array}}, \quad B = \frac{\begin{array}{c|c|c|c|c} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}}{\begin{array}{c|c|c|c|c} 0{,}5 & 0 & 0 & 0{,}2 & 0{,}3 \end{array}}.$$

According to Definition 4.14

$x_i \in B$ / $x_i \in A$	x_1	x_2	x_3	x_4	x_5
x_1	0,1	0	0	0,1	0,1
x_2	0,5	0	0	0,2	0,3
x_3	0,3	0	0	0,2	0,3
x_4	0,5	0	0	0,2	0,3
x_5	0,2	0	0	0,2	0,2

$A \times B =$ (the table above)

Definition 4.15. *A convex combination* of fuzzy subsets A_1, ..., A_n of a universal set X is a fuzzy set A with membership function, has the following form:

$$\mu_A(x) = \sum_{i=1}^{n} \lambda_i \mu_i(x), \tag{4.53}$$

where $\lambda_i \ge 0$, $i = 1,..,n$, $\sum_{i=1}^{n} \lambda_i = 1$.

With respect to classical sets, in contrast to a Cartesian product, the operation of a convex combination does not make sense.

Definition 4.16. *Concentration* operations (CON) and *dilatation* (DIL) operations are defined as follows:

$$\text{CON}\, A = A^2, \tag{4.54}$$

$$\text{DIL}\, A = A^{0,5}, \tag{4.55}$$

wherein

$$\mu_{A^\alpha}(x) = \mu_A^\alpha(x),\ x \in X,\ \alpha > 0. \tag{4.56}$$

Example 4.31. Let the universal set $E = \{x_1, \dots, x_n\}$, $A \subset E$,

$$A = \frac{x_1 \mid x_2 \mid x_3 \mid x_4 \mid x_5 \mid x_6}{0{,}25 \mid 0{,}9 \mid 0{,}4 \mid 0{,}6 \mid 1 \mid 0}.$$

Define the sets: $B = \text{CON}\, A$, $C = \text{DIL}\, A$, namely:

$$B = \frac{x_1 \mid x_2 \mid x_3 \mid x_4 \mid x_5 \mid x_6}{0{,}0625 \mid 0{,}81 \mid 0{,}16 \mid 0{,}36 \mid 1 \mid 0},$$

$$C = \frac{x_1 \mid x_2 \mid x_3 \mid x_4 \mid x_5 \mid x_6}{0{,}5 \mid 0{,}95 \mid 0{,}63 \mid 0{,}77 \mid 1 \mid 0}.$$

Example 4.32. Let the fuzzy set $A \subset R^1$ be fed by its membership function

$$\mu_A(x) = \frac{1}{1+|x-a|}, \text{ then } \mu_{A^2}(x) = \frac{1}{(1+|x-a|)^2}.$$

Graphically, these sets can be represented as follows:

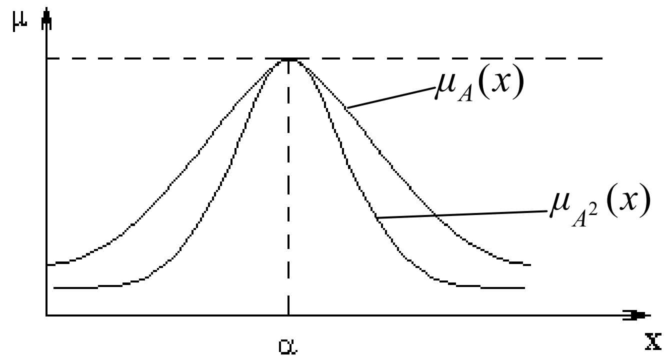


Fig. 4.13. Graphs of membership functions of fuzzy sets *A* and CON *A*.

Applying the concentration operation to a fuzzy set means reducing its "fuzziness". In real problems, this can mean the arrival of new information that allows more accurately (clearly) describe the presented fuzzy set. Similarly, the stretch operation can be used to simulate situations involving loss of information.

4.8 Fuzzy relations

Definition 4.17. A fuzzy relation *R* in a set *X* is referred to as a fuzzy subset of the Cartesian product $X \times X$. It is characterized by such membership function $\mu_R : X \times X \to [0;1]$.

The value $\mu_R(x, y)$ of this function indicates the degree which relation *R* between the elements *x* and *y* is satisfied. It is clear that we can regard classical relations as a separate case of fuzzy relations, the functions of membership to which can have only two values: 0 or 1.

Example 4.32. We consider two similar relations on the interval [0; 1]. This is the classical relation of "greater than or equal to" *R* ($\geq$) and a fuzzy relation "*significantly larger*" $\widetilde{R}$ $(>>)$. The pairs connected by the relation *R* are shown in Figure 4.14, and the relation $\widetilde{R}$ is shown in Figure 4.15.

If there is a fuzzy relation *R*, then there are pairs of elements for which it is performed clearly, there are pairs for which this relation is not satisfied, and also there is some intermediate zone where the pairs have this or that degree of membership, that is, the relation is satisfied for them only at a certain degree depending on the situation. The fuzzy border in this case is represented by the variable hatching density.

Just as with normal relations (see Section 2), fuzzy relations can be defined by a matrix, graph or cuts.

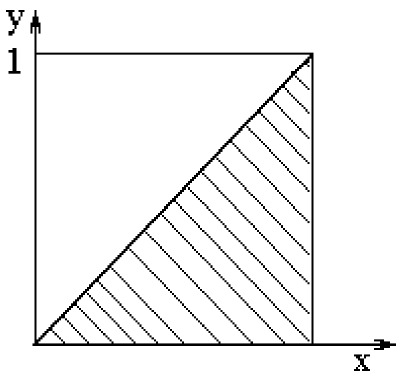


Fig. 4.14. Graphic representation of the relation «≥»

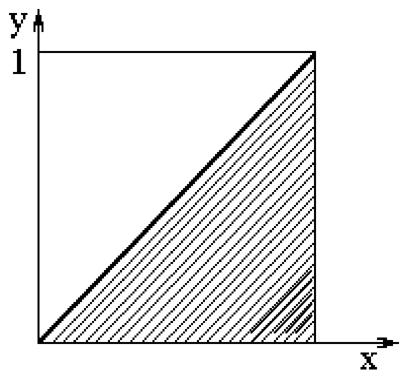


Fig. 4.15. Graphic representation of the relation «>>»

A matrix of a fuzzy relation is similar to the matrix of the classical relation, only its elements can be numbered from 0 to 1.

When a fuzzy relation is specified with a graph, each arc is assigned a number from the interval [0; 1], which means the degree of fulfillment of the fuzzy relation for a given pair.

The upper and lower sections of the fuzzy relation are fuzzy sets defined as follows:

$$R^{+}(x)=\{y|\mu_R(y,x):\ y\in X,\ \mu_R(y,x)>0\},$$

$$R^{-}(x)=\{y|\mu_R(x,y):\ y\in X,\ \mu_R(x,y)>0\}.$$

Definition 4.18. A *support* of a fuzzy relation *R* in a set *X* is a subset of the Cartesian product $X \times X$ which looks like this:

$$\text{supp}\, R=\{(x,y)|(x,y)\in X\times X,\ \mu_R(x,y)>0\}.$$

A fuzzy relation support can be understood as a classical relation in the set *X*, which connects the pairs (x,y) for which the relation R is satisfied from a non-zero power.

Example 4.33. Let the relation *R* be "approximately equal". We define it in the set: $X=\{1,2,3,4,5\}$ using the matrix. It can have the following form:

$$R = \begin{vmatrix} 1 & 0,5 & 0,2 & 0 & 0 \\ 0,5 & 1 & 0,5 & 0,2 & 0 \\ 0,2 & 0,5 & 1 & 0,5 & 0,2 \\ 0 & 0,2 & 0,5 & 1 & 0,5 \\ 0 & 0 & 0,2 & 0,5 & 1 \end{vmatrix}.$$

Then, the support of a described fuzzy relation will be the following classical relation:

$$\operatorname{supp} R = \begin{vmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{vmatrix}.$$

Note that the specific form of the matrix of relation depends on the meaning of the problem and how the expression "approximately equal" is understood.

4.9 Operations on fuzzy relations

In the previous subsections, operations on fuzzy sets and classical relations were considered. Operations on fuzzy relations combine the properties of both relations to some extent. In other words, some of them are analogues of the corresponding operations with classical relations. But there are also those that belong only to fuzzy relations. For example, operations of combining and intersecting fuzzy relations can be defined in the same ways as for fuzzy sets.

Definition 4.19. Suppose two fuzzy relations A and B are given in the set X, that is, two fuzzy subsets A and B are presented in the Cartesian product X^2. Then the fuzzy sets: $C = A \cap B$ and $D = A \cup B$, will be called respectively the *intersection* and *union* of the fuzzy relations A and B on the set X.

Example 4.34. relations A and B are represented in this form

$$A = \begin{vmatrix} 0,5 & 1 & 0 \\ 0 & 0,1 & 0,8 \\ 1 & 1 & 0,4 \end{vmatrix}, \qquad B = \begin{vmatrix} 1 & 0,2 & 0 \\ 0,3 & 1 & 1 \\ 0,8 & 0,8 & 1 \end{vmatrix}.$$

Let us find an intersection and a union of these relations, using Definitions 4.6 and 4.7, namely:

$$A \cap B = \begin{vmatrix} 0{,}5 & 0{,}2 & 0 \\ 0 & 0{,}1 & 0{,}8 \\ 0{,}8 & 0{,}8 & 0{,}4 \end{vmatrix}, \qquad A \cup B = \begin{vmatrix} 1 & 1 & 0 \\ 0{,}3 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}.$$

Definition 4.20. A fuzzy relation B *contains* a fuzzy relation A if for fuzzy sets B and A *t*here is an inclusion: $A \subset B$, their membership functions satisfy the following inequality:

$$\mu_A(x, y) \leq \mu_B(x, y), \quad \forall x, y \in X .$$

For example, in example 4.32 above, the relation "≥" contains the relation ">>".

Definition 4.21. If R is a fuzzy relation in the set X, then the fuzzy relation $\overline{R}$ which membership function $\mu_{\overline{R}}(x, y) = 1 - \mu_R(x, y)$ we call the *complement* of relation R of the set X.

For example, the complement of a fuzzy relation "*better*" would be "*no better*".

The fuzzy relation R^{-1} in the set X that is *inverse* to R is defined as follows:

$$x\, R^{-1}\, y \;\Leftrightarrow\; y\, R\, x, \quad \forall x, y \in X ,$$

or using the terminology of membership functions

$$\mu_{R^{-1}}(x, y) = \mu_R(y, x), \quad \forall x, y \in X.$$

Unlike classical relations, the product (or composition) of fuzzy relations can be defined in many ways.

Let`s consider some of the possible definitions of this operation.

Definition 4.22. The *maximin* product of fuzzy relations A and B in the set X is described by such a membership function:

$$\mu_{A \cdot B}(x, y) = \sup_{z \in X} \min\{\mu_A(x, z),\ \mu_B(z, y)\}.$$

If the output relations are given in a finite set X, then the matrix of their maximin product is equal to the maximin product of the matrices of the relations A and B.

Definition 4.23. The *minimax* product of fuzzy relations A and B in the set X will be equal to the fuzzy relation which membership function is

$$\mu_{A \cdot B}(x, y) = \inf_{z \in X} \max\{\mu_A(x, z),\ \mu_B(z, y)\}.$$

Definition 4.24. The *maximal multiplicative* product of fuzzy relations A and B is characterized by the membership function of the following form:

$$\mu_{A \cdot B}(x, y) = \sup_{z \in X} \{ \mu_A(x, z) \cdot \mu_B(x, z) \}.$$

Example 4.34. Let the fuzzy relations A and B be given with the help of matrices:

$$A = \begin{vmatrix} 1 & 0 & 0,5 & 0,2 \\ 0,7 & 1 & 0,8 & 0 \\ 0,3 & 0,4 & 0,3 & 0 \\ 0 & 0,2 & 0,8 & 1 \end{vmatrix}, \qquad B = \begin{vmatrix} 0,5 & 0 & 1 & 0 \\ 0,2 & 0,3 & 1 & 0 \\ 1 & 0,3 & 0,5 & 0 \\ 1 & 1 & 0 & 0,5 \end{vmatrix}.$$

Let's find the composition of the relations A and B, using the definitions 4.22 – 4.24, then, we get the following results:

the maximin composition will be described by such a matrix:

$$A \cdot B_{\max\min} = \begin{vmatrix} 0,5 & 0,3 & 1 & 0,2 \\ 0,8 & 0,3 & 1 & 0 \\ 0,3 & 0,3 & 0,4 & 0 \\ 1 & 1 & 0,5 & 0,5 \end{vmatrix},$$

minimax composition – matrix of the following form:

$$A \cdot B_{\min\max} = \begin{vmatrix} 0,2 & 0,3 & 0,2 & 0 \\ 0,7 & 0.7 & 0 & 0,5 \\ 0,4 & 0,3 & 0 & 0,3 \\ 0,2 & 0 & 0,8 & 0 \end{vmatrix},$$

and the maximal multiplicative composition is such a matrix:

$$A \cdot B_{\max\cdot} = \begin{vmatrix} 0,5 & 0,2 & 1 & 0,1 \\ 0,8 & 0,3 & 1 & 0 \\ 0,3 & 0,12 & 0,4 & 0 \\ 1 & 1 & 0,4 & 0,5 \end{vmatrix}.$$

4.10 Properties of fuzzy relations

Let's consider now what are the characteristics of fuzzy relations.

Definition 4.25. The fuzzy relation *R* in the set *X* is called *reflexive* if for any elements $x \in X$ the following condition is fulfilled:

$$\mu_R(x,x) = 1.$$

If the reflexive relation is given by a matrix, then its main diagonal includes only units.

An example of a reflexive relation is "*approximately equal*" is given by a set of numbers.

Definition 4.26. Fuzzy relation *R* will be antireflexive if $\mu_R(x,x) = 0$, $\forall x \in X$.

The complement of the reflexive relation will be antireflexive. An example of an antireflexive set of numbers can be the relation "*much more*".

Definition 4.27. The fuzzy relation *R* in the set *X* is called *symmetric* if for any elements $x, y \in X$, the following condition is fulfilled:

$$\mu_R(x,y) = \mu_R(y,x).$$

The matrix of a symmetric fuzzy relation given in a finite set will be symmetric. This example will be the relation "very different in size."

Definition 4.28. The relation *R* on the set *X* will be *asymmetric* if it has the following property:

$$\mu_R(x,y) > 0, \Rightarrow \mu_{\mathrm{R}}(y,x) = 0, \quad \text{або} \quad \mu_R(x,y) = \mu_R(y,x) = 0, \quad \forall x, y \in X.$$

in other words:

$$\min\{\mu_R(x,y), \mu_R(y,x)\} = 0, \forall x, y \in X.$$

Asymmetric is the relation of "much more" type.

Definition 4.29. The relation *R* oi the set *X* will be *antisymmetric* if the following condition holds:

$$\min\{\mu_R(x,y), \mu_R(y,x)\} = 0, \quad x \neq y.$$

Definition 4.30. The fuzzy relation R on the set *X* is called *transitive*, if $R^2 \subset R$.

Obviously, the property of transitivity depends on the method of determining the product of relations. According to the earlier definitions it is possible to name its three types: *maximin* (max min), *minimax* (min max) and *maxmultiplicative* (max ·) transitivity.

It is easy to see that $R^2_{\max\text{-}\bullet} \subseteq R^2_{\max\min}$. Thus, max min of transitivity leads to maximal multiplicative transitivity.

An example of max min-transitive is the relation "*much more*" in the set of numbers.

Example 4.35. Check for transitivity a fuzzy relation that has the following form:

$$R = \begin{vmatrix} 1 & 0 & 0 & 0 & 0{,}4 \\ 0 & 1 & 0{,}9 & 1 & 0 \\ 0 & 1 & 0{,}9 & 1 & 0 \\ 0 & 1 & 0{,}9 & 1 & 0 \\ 0{,}4 & 0 & 0 & 0 & 1 \end{vmatrix}.$$

Solution

To check this property, it is necessary to calculate max min, min max and max multiplicative compositions of this relation.

$$R^2_{\max\min} = \begin{vmatrix} 1 & 0 & 0 & 0 & 0{,}4 \\ 0 & 1 & 0{,}9 & 1 & 0 \\ 0 & 0{,}9 & 0{,}9 & 0{,}9 & 0 \\ 0 & 1 & 0{,}9 & 1 & 0 \\ 0{,}4 & 0 & 0 & 0 & 1 \end{vmatrix}.$$

Since $R^2_{\max\min} \subset R$, then, the fuzzy relation R is max min transitive. Therefore, it will be both multiplicative and transitive. Let's check the relation for min max transitivity. The corresponding composition

$$R^2_{\min\max} = \begin{vmatrix} 0 & 0{,}4 & 0{,}4 & 0{,}4 & 0 \\ 0{,}4 & 0 & 0 & 0 & 0{,}4 \\ 0{,}4 & 0 & 0 & 0 & 0{,}4 \\ 0{,}4 & 0 & 0 & 0 & 0{,}4 \\ 0 & 0{,}4 & 0{,}4 & 0{,}4 & 0 \end{vmatrix}.$$

As you can see, $R^2_{\max\min} \not\subset R$, consequently, the relation R will not be min max-transitive.

Definition 4.31. The *transitive closure* of the fuzzy relation R will be a fuzzy relation $\hat{R}$, which is obtained by the following rule:

$$\hat{R} = R \cup R^2 \cup R \cup \ldots \cup R^n \cup \ldots$$

Obviously, when determining the transitive closure, it is necessary first to establish the type of operation for a product of relations.

With regard to a transitive closure, the following statement applies:

Theorem 4.2. A transitive closure of any binary relation R is the least transitive binary relation that contains R.

Note that the α-level of the transitive closure of a fuzzy relation coincides with the transitive closure of the corresponding α-level of the source fuzzy relation, that is,

$$\left(\hat{R}\right)_\alpha = \hat{R}_\alpha, \quad \forall \alpha \neq 0.$$

Here is a formulation of two theorems that allow us to construct a transitive closure in some cases.

Theorem 4.3. If there is a number k, for which $R^k = R^{k+1}$, then

$$\hat{R} = R \cup R^2 \cup \ldots \cup R^k .$$

Theorem 4.4. If R is a fuzzy relation in a finite set E, then $m(E) = n$, then $\hat{R} = R \cup R^2 \cup R \cup \ldots \cup R^n$ or there is a number: $k \leq n$ for which $R^k = R^{k+1}$.

Example 4.36. We construct a transitive (max min) closure of the fuzzy relation R given by a matrix:

$$R = \begin{vmatrix} 0{,}8 & 1 & 0{,}1 \\ 0 & 0{,}4 & 0 \\ 0{,}3 & 0 & 0{,}2 \end{vmatrix}.$$

To do this, we will calculate successively R^2, R^3, namely:

$$R^2_{\max\min} = \begin{vmatrix} 0{,}8 & 0{,}8 & 0{,}1 \\ 0 & 0{,}4 & 0 \\ 0{,}3 & 0{,}3 & 0{,}2 \end{vmatrix}, \qquad R^3_{\max\min} = \begin{vmatrix} 0{,}8 & 0{,}8 & 0{,}1 \\ 0 & 0{,}4 & 0 \\ 0{,}3 & 0{,}3 & 0{,}2 \end{vmatrix}.$$

We see that $R^2 = R^3$ consequently, $\hat{R} = R \cup R^2$ and takes the following form:

$$\hat{R} = \begin{vmatrix} 0 & 1 & 0{,}1 \\ 0 & 0{,}4 & 0 \\ 0{,}3 & 0 & 0{,}2 \end{vmatrix} \cup \begin{vmatrix} 0{,}8 & 0{,}8 & 0{,}1 \\ 0 & 0{,}4 & 0 \\ 0{,}3 & 0{,}3 & 0{,}2 \end{vmatrix} = \begin{vmatrix} 0{,}8 & 1 & 0{,}1 \\ 0 & 0{,}4 & 0 \\ 0{,}3 & 0{,}3 & 0{,}2 \end{vmatrix}.$$

4.11 Classification of fuzzy relations

Taking into account fuzzy relation properties, all types of fuzzy relations can be divided into three classes.

The first class includes symmetric relations, the majority of which are characterized by the similarity or difference between objects of a set X. Such relations can be set by a weighted graph with non-oriented arcs.

The second class is formed by antisymmetric relations. They specify ranking dominating relations in a set. They are matched with oriented weighted graphs with one-sided orientation of arcs.

The third class includes the remaining relations.

The relations of each class in their turn can be subdivided into subclasses from the perspective of fulfilling the conditions of reflexivity or antireflexivity. Schematically, the classification of fuzzy relations is presented in Figure 4.14, a more detailed classification can be found in the collection [25]. Let's consider some of the fuzzy relations.

The fuzzy relation *of a pre-order* is referred to a binary fuzzy relation that has the properties of transitivity and reflexivity.

If R is a pre-order, then there is the following equality:

$$R = R^2 = ... = R^k = \hat{R}.$$

The pre-order in the set: $X = \{A, B, C, D, E\}$, for example, will be a relation that looks like this:

$$\begin{vmatrix} 1 & 0{,}7 & 0{,}8 & 0{,}5 & 0{,}5 \\ 0 & 1 & 0{,}3 & 0 & 0{,}5 \\ 0 & 0{,}7 & 1 & 0 & 0{,}2 \\ 0{,}6 & 1 & 0{,}9 & 1 & 0{,}6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}.$$

Fuzzy half-order is a transitive relation that does not have the properties of reflexivity.

Symmetrical, reflexive relations are called relation*s of similarities*. They show the degree of similarity ("proximity") of two elements.

Symmetric, anti-reflexive relations are called relations of differences.

For the relations of similarities and differences are characterized by the following assertion the statement: if R is a fuzzy relation of similarity, then $\overline{R}$ the relation of difference.

Among the close relations special attention is drawn to similarity itself.

Classification of fuzzy relations

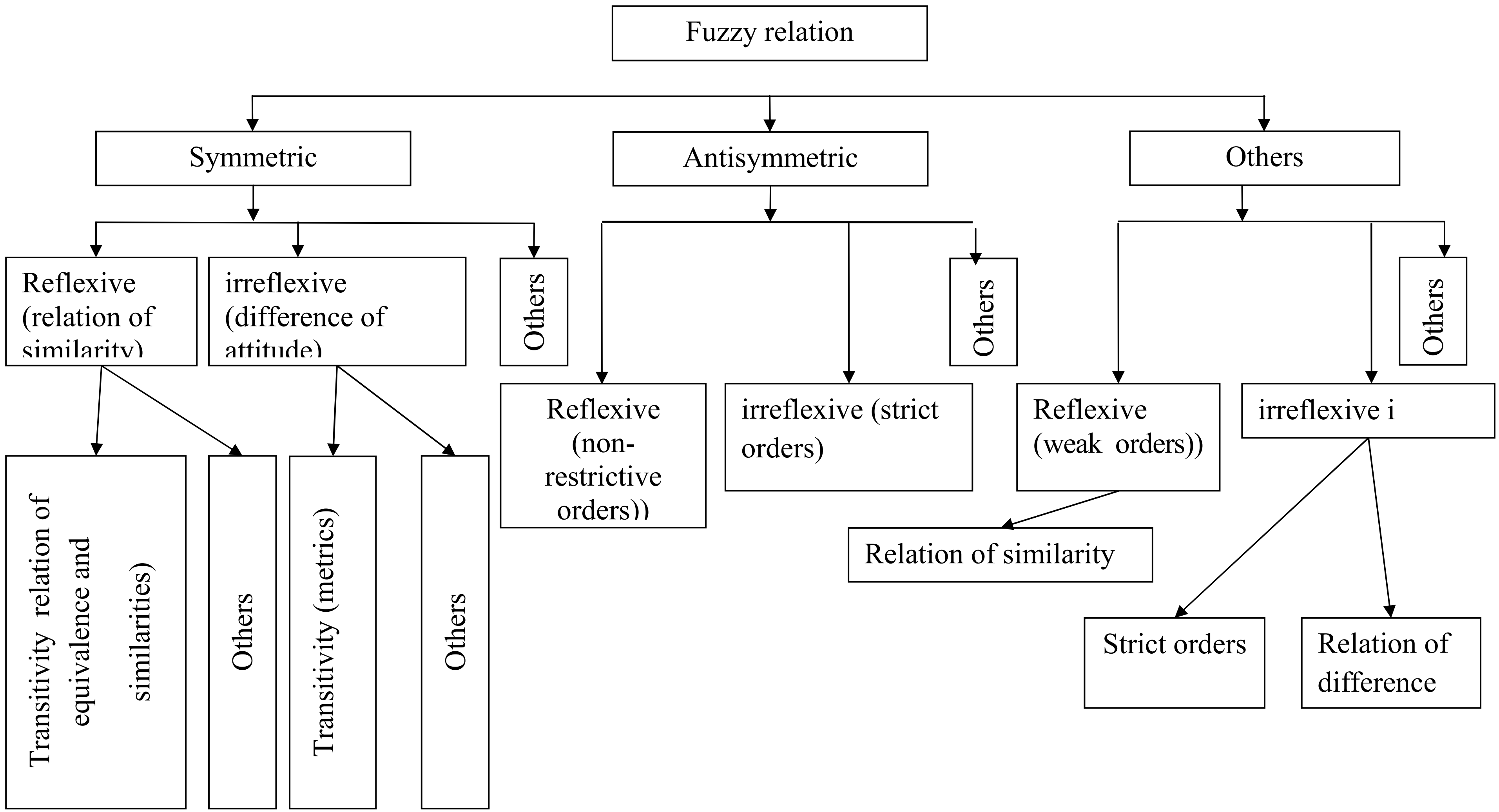


Fig. 4.14. Scheme of the relationship between types of fuzzy relations

Definition 4.32. The relation of similarity or equivalence is called a fuzzy binary relation, which is characterized by transitivity, reflexivity and symmetry.

Obviously, this relation is a pre-order one.

Relation

$$R_1 = \begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & 1 & 0{,}8 & 0{,}7 & 1 & 0{,}9 \\ B & 0{,}8 & 1 & 0{,}7 & 0{,}8 & 0{,}8 \\ C & 0{,}7 & 0{,}7 & 1 & 0{,}7 & 0{,}7 \\ D & 1 & 0{,}8 & 0{,}7 & 1 & 0{,}9 \\ E & 0{,}9 & 0{,}8 & 0{,}7 & 0{,}9 & 1 \end{array}$$

is a fuzzy relation of similarity.

General, the relation of this kind:

$$R_2 = \begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & 1 & a & a & a & a \\ B & a & 1 & a & a & a \\ C & a & a & 1 & a & a \\ D & a & a & a & 1 & a \\ E & a & a & a & a & 1 \end{array}, \quad \text{if } a \in [0,1],$$

will be similarities.

Task is to check the transitivity of these relations.

Example 4.40. Fuzzy relation $x\,R\,y$, where $x, y \in [0;+\infty)$, defined by such membership function:

$$\mu_R(x,y) = \begin{cases} e^{-k(y+1)}, & y < x,\ k > 1, \\ 1, & y = x, \\ e^{-k(x+1)}, & y > x,\ k > 1. \end{cases}$$

Task*:* Check it out for yourself whether it is related to a relation of similarity.

Each α-level of fuzzy relation of similarity is a normal equivalence relation. We would like to remind that any relation of equivalence makes some partition of a set. Consequently, each α-level of fuzzy relations of similarity will also make partitions in this set. The property of α-levels of fuzzy relation implies the corresponding partitions of the set *X*. Moreover, with the decrease of α, the integration of the equivalence classes occurs. Thus, the unclear equivalence relation, in contrast to the classical relation of similarity, implies hierarchical set of distributions on non-intersecting equivalence classes in the set *X*. This is explained by the fact that the condition of transitivity imposes rather strong restrictions to the degree of membership $\mu(x, y)$. Namely, for the fuzzy relation of similarity, the following theorem works.

Theorem 4.5. Let $R \subset E \times E$ is a relation of similarity and *x, y, z* are three elements of the set *E*. We prescribe that

$$c = \mu_R(x, z) = \mu_R(z, x),$$

$$a = \mu_R(x, y) = \mu_R(y, x),$$

$$c = \mu_R(y, z) = \mu_R(z, y).$$

Then, $c \geq a = b$ or $a \geq b = c$, or $b \geq c = a$, that is, at least two values from *a*, *b*, *c* are equal, i.e. equivalent, and the third value is more than them.

Definition 4.33. The fuzzy binary relation, which has the properties of antireflexivity and symmetry, is called the relation of difference. Examples of relation of difference are:

1. The relation of the set {*A*, *B*, *C*, *D*, *E*} is given by a matrix of the following form:

	A	*B*	*C*	*D*	*E*
A	0	0,2	0,3	0	0,1
B	0,2	0	0,3	0,2	0,2
C	0,3	0,3	0	0,3	0,3
D	0	0,2	0,3	0	0,1
E	0,1	0,2	0,3	0,1	0

2. Fuzzy relation given by such membership function:

$$\mu_{\overline{R}}(x, y) = \begin{cases} 1 - e^{-k(y+1)}, & y < x, \quad k > 1, \\ 0, & y = x, \\ 1 - e^{-k(x+1)}, & y > x, \quad k > 1, \end{cases}$$

is a relation of differences. It is formed as a result of substitution: $\mu_{\overline{R}}(x, y) = 1 - \mu_R(x, y)$ in Example 4.40.

The measure of difference can be considered as the distance between the elements of the set (if adding transitivity). Moreover, different types of transitivity, respectively, assign different types of distances.

Definition 4.34. The fuzzy relation *of order* is referred to the binary characterized by the properties of reflexivity, transitivity and anti-symmetry.

It is distinguished relations of strict and non-strict order.

Strict order is an antireflexive, asymmetric and transitive relation.

Relations of non-strict order are reflexive, antisymmetric and transitive.

4.12 Fuzzy set mapping. Generalization principle

In many decision-making tasks, there is a need to extend the domain of definition *X* of the given mapping or relation by including along with the individual elements of the set *X*, to its arbitrary fuzzy subsets.

For example, a set of controls (guiding or managing influences) *U* has a mapping $f: U \to V$ that describes the functioning of a managed system. For each control $u \in U$ its image: $v = f(u)$, displays the response of this system to the choice of this control. If the selected control is not clearly described, for example, in the form of a fuzzy subset $\mu(u)$ of *U*, then to find the system response to it, it is necessary to define the image $\mu(u)$ when f is mapped.

The method of extending the scope of defining image for a class of fuzzy sets is called the *generalization principle*.

L. A. Zadeh suggested the generalization principle, which is based on determining the image of a fuzzy set under the classical (clearly described) representation.

Let the map $\varphi: X \to Y$, and *A* be a subset of the set *X*, which is characterized by a membership function $\mu_A(x)$.

Definition 4.35. In the image of a fuzzy set *A* when displayed with *φ*, we will call the fuzzy subset of the set *Y* as a combination of such pairs:

$$(y, \mu_B(y)) = (\varphi(x), \mu_A(x)),\ x \in X,$$

where μ_B - the function of the property of the image, $\mu_B: Y \to [0;1]$.

It is easy to understand that the membership function μ_B can be written as follows:

$$\mu_B(y) = \sup_{x \in \varphi^{-1}(y)} \mu_A(x),\ y \in Y, \tag{4.57}$$

and the set $\varphi^{-1}(y)$ for each fixed element $y \in Y$ is determined by the following rule:

$$\varphi^{-1}(y) = \{x \mid x \in X, \varphi(\mathrm{x}) = y\},$$

that is, a set of all those elements $x \in X$, the image of which when the mapping φ will be *y*.

Example 4.42. Let the set $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, and the set $Y = \{y_1, y_2, y_3\}$. The mapping $\varphi : X \to Y$ is given by the table:

	y_1	y_2	y_3
x_1	1	0	0
x_2	0	1	0
x_3	0	1	0
x_4	1	0	0
x_5	0	0	1
x_6	1	0	0
x_7	0	0	1

In the set *X* we define a fuzzy subset *A*:

$$A = \begin{array}{c|c|c|c|c|c|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline 0{,}3 & 0{,}5 & 1 & 0 & 0{,}5 & 0{,}8 & 0{,}4 \end{array}$$

We find an image *B* in a fuzzy subset of *A* when mapped with φ. In accordance with definition 4.35

$$\mu_B(y_1) = \sup_{x \in \varphi^{-1}(y_1)} \mu_A(x),$$

$$\varphi^{-1}(y_1) = \{x_1, x_4, x_6\},$$

then $\mu_B(y_1) = \sup\limits_{x \in \{x_1, x_4, x_6\}} \mu_A(x) = \sup\{0{,}3;\ 0;\ 0{,}8\} = 0{,}8$.

Similarly

$$\mu_B(y_2) = \sup_{x \in \varphi^{-1}(y_2)} \mu_A(x) = \sup_{x \in \{x_2, x_3\}} \mu_A(x) = \sup\{0{,}5;\ 1\} = 1,$$

$$\mu_B(y_3) = \sup_{x \in \varphi^{-1}(y_3)} \mu_A(x) = \sup_{x \in \{x_5, x_7\}} \mu_A(x) = \sup\{0{,}5;\ 0{,}4\} = 0{,}5.$$

Thus, the image of the set *A* in the representation φ will be a fuzzy set $B \subset Y$, which looks like this:

$$B = \begin{array}{c|c|c} y_1 & y_2 & y_3 \\ \hline 0{,}8 & 1 & 0{,}5 \end{array}.$$

We now apply the generalization principle to expand the area of fuzzy mapping.

The mapping of the set X to the set Y is called *fuzzy* if each element $x \in X$ is compared with a corresponding certain fuzzy subset of the set Y. A fuzzy representation of the membership function is described as $\mu_\varphi : X \times Y \to [0;1]$, wherein the function $\mu_\varphi(x_0, y)$ for each fixed element: $x = x_0$ is a membership function of a fuzzy subset in the set Y, which is the image of the element x_0 with the mapping φ.

Consequently, let it be given a fuzzy mapping $\mu_\varphi : X \times Y \to [0;1]$ and $\mu_A(x)$ fuzzy subset of the set X. If for the purpose of finding the image of this fuzzy set at the mapping μ_φ we apply the generalization principle in accordance with rule (4.57), then we obtain the following set of pairs:

$$(\mu_\varphi(x, y), \mu_A(x)), \quad x \in X,$$

where the function $\mu_\varphi(x, y)$ for each fixed element x defines a fuzzy subset of the set Y.

As a result, we can conclude that the image of the fuzzy set $\mu_A(x)$ in this case is a sufficiently complex object, namely, a fuzzy subclass of all fuzzy subsets of the set Y. Consequently, there is a need to introduce the principle of generalization in a different form.

Definition 4.36. In the image B of a fuzzy set $A \subset X$ with fuzzy mapping $\mu_\varphi : X \times Y \to [0;1]$ is referred to a fuzzy subset of the set Y characterized by the following membership function:

$$\mu_B(y) = \sup_{x \in X} \min\{\mu_A(x), \mu_\varphi(x, y)\}. \tag{4.58}$$

The basis of this definition is the maximizing product (composition) of fuzzy relations.

When φ is the classical mapping, that is $\mu_\varphi(x, y) = 1$, if $y = \varphi(x)$, then formula (4.58) turns into (4.57).

In many tasks, the initial fuzzy mapping μ_φ depends on n variables, that is, has the following form: $\mu_\varphi : X \times Y \to [0;1]$, where $X = X_1 \times X_2 \times \ldots \times X_n$.

Assume that a fuzzy subset μ_A is given in the set X. In general case, the membership function of this subset has the following form:

$$\mu_A(x_1,...,x_n) = \min(\mu_1(x_1),..., \mu_n(x_n); \nu(x_1,..., x_n)),$$

Here $\mu_i(x)$, $i=1,\dots n$ and $\nu(x_1 \dots x_n)$ are known membership functions of fuzzy subsets of the set X_i, $i=1,\dots n$ and X, respectively.

Applying the generalization principle in accordance with rule (4.58) for this case, we obtain the following formula for the membership function of the image of a fuzzy subset μ_A:

$$\mu_B(y) = \sup_{(x_1,...,x_n)\in X} \min\{\mu_1(x_1),..., \mu_n(x_n), \nu(x_1,..., x_n), \mu_\varphi(x_1,..., x_n, y)\}. \quad (4.59)$$

Example 4.42. We have sets: $X = \{x_1,..., x_7\}$, $Y = \{y_1, y_2, y_3\}$ and a fuzzy subset *A* of the set *X*, which is given as follows:

$$A = \frac{x_1 \mid x_2 \mid x_3 \mid x_4 \mid x_5 \mid x_6 \mid x_7}{0{,}5 \mid 0{,}8 \mid 1 \mid 0 \mid 0{,}1 \mid 0{,}4 \mid 1}.$$

Besides, we know the fuzzy mapping $\varphi : X \to Y$ which membership function $\mu_\varphi : X \times Y \to [0;1]$ is given by a table, i.e.

$$\mu_\varphi = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0{,}2 & 0{,}1 & 0 \\ x_2 & 0{,}4 & 0{,}8 & 1 \\ x_3 & 1 & 0{,}1 & 0 \\ x_4 & 0 & 0{,}5 & 1 \\ x_5 & 0 & 0 & 1 \\ x_6 & 0{,}3 & 0 & 0{,}8 \\ x_7 & 0{,}9 & 0{,}3 & 0{,}4 \end{array}$$

It is necessary to determine the image $B \subset Y$ of the set *A* with fuzzy mapping φ.

Solution

We will apply definition 4.36. Then, to calculate the membership function of the set *B,* we use the maximization product of the functions μ_A and μ_φ. The obtained results are presented below.

$$(0,5;\ 0,8;\ 1;\ 0;\ 0,1;\ 0,4;\ 1)\cdot\begin{vmatrix}0,2 & 0,1 & 0\\ 0,4 & 0,8 & 1\\ 1 & 0,1 & 0\\ 0 & 0,5 & 1\\ 0 & 0 & 1\\ 0,3 & 0 & 0,8\\ 0,9 & 0,3 & 0,4\end{vmatrix}=(1;\ 0,8;\ 0,8).$$

So, $B=\dfrac{y_1 \mid y_2 \mid y_3}{1 \mid 0,8 \mid 0,8}$.

Example 4.43. We extend the scope of defining the arithmetic operation of adding to the class of "fuzzy numbers", i.e. the class of fuzzy subsets of the numerical axis.

The complement operation in the set of numbers is a mapping $\varphi: R^1\times R^1\to R^1$, i.e. $\varphi(r_1,r_2)=r=r_1+r_2$.

Assume that μ_1,μ_2 are two fuzzy numbers ($\mu_1,\mu_2: R^1\to[0,1]$). The image of a pair (μ_1,μ_2) when displaying φ, we will call them the sum: $\mu_\Sigma=\mu_1+\mu_2$. Then, using formula (4.59), we obtain the following result:

$$\mu_\Sigma(r)=\sup_{\substack{r_1,r_2\in R^1\\ r_1+r_2=r}}\min\{\mu_1(r_1),\mu_2(r_2)\}. \tag{4.60}$$

In particular, when fuzzy numbers μ_1 and μ_2 are intervals $[a_1,b_1]$ and $[a_2,b_2]$ then, according to formula (4.60) $[a_1,b_1]+[a_2,b_2]=[a_1+a_2,\ b_1+b_2]$.

Definition 4.37. The prototype *A* of a fuzzy set $B\subset Y$ with a fuzzy mapping $\mu_\varphi: X\times Y\to[0;1]$ is called the union of all fuzzy sets, the images of which at this mapping belong to the fuzzy set *B*, i.e. they are subsets of *B*.

Let's denote the image of the fuzzy set μ_φ through $A\cdot\mu_\varphi$. Then, the condition for determining the pre-image of a set can be written as follows:

$$\sup_{x\in X}\min\{\mu_2(x),\mu_\varphi(x,y)\}\le\mu_B(y),\ \forall y\in Y. \tag{4.61}$$

It comes from inclusion $A\cdot\mu_\varphi\subset Y$.

The explicit expression for the prototype membership function is given by the following theorem. For its wording, we will give the following sets:

$$N = \{(x, y) | (x, y) \in X \times Y, \mu_\varphi(x, y) > \mu_B(y)\},$$
$$N_x = \{y | y \in Y, \ (x, y) \in N\},$$
$$N_y = \{x | x \in X, \ (x, y) \in N\},$$
$$X^\circ = \{x | x \in X, N_x \neq \varnothing\}.$$

Theorem 4.6. In the given above notation, the fuzzy set *A* (the prototype of the set *B*) is described by the following membership function:

$$\mu_A(x) = \begin{cases} \inf\limits_{y \in N_x} \mu_B(y), \ x \in X^\circ, \\ 1, x \in X \setminus X^\circ. \end{cases}$$

It is easy to check that when image μ_φ is clear, that is φ a normal display $\varphi : X \to Y$ and membership function

$$\mu_\varphi(x, y) = \begin{cases} 1, \ \text{if } y = \varphi(x), \\ 0 \ \text{ for all other couples } (x, y) \in X \times Y, \end{cases}$$

then, $\mu_A(x) = \mu_B(\varphi(x)), \quad \forall x \in X$.

Conclusions

Fuzzy sets act as a generalization of the notion of a classical set in cases where an element can belong to a set only to a certain extent. The theory of fuzzy sets allows us to describe situations of uncertainty more adequately due to the impossibility of clearly to describe the preferences or the set of permissible alternatives.

Operations over fuzzy sets can be defined in different ways, depending on specific tasks, provided that they are performed correctly in relation to distinct sets.

Fuzzy relations are an extension of the concept of binary relation to a class of fuzzy sets. Their properties are substantiated by features of fuzzy sets and binary relations.

The generalization principle is a way of expanding the scope of mapping to a class of fuzzy sets.

SELF-STUDY

Questions for assessment and self-assessment

1. What does the characteristic feature of the set mean?
2. Give a definition of a fuzzy set.
3. What is called a fuzzy set support?
4. What operations about fuzzy sets do you know?
5. How can be fuzzy set complements defined? Union and intersection of fuzzy sets?
6. What explains the existence of several operations of union and intersection of fuzzy sets?
7. What special operations with fuzzy sets do you know?
8. What is the sense of concentration and dilatation operations?
9. How is the Hamming distance calculated when considering a finite set? Counting set? Set of continuum cardinality?
10. How do we calculate the Euclidean distance between sets?
11. What is the geometric meaning of the linear distance between sets?
12. What property characterizes the index of fuzzy sets? How is it calculated?
13. Does the index of fuzziness of the intersection (union) of sets depend on the indices of fuzzy input sets?
14. Does the index of fuzzy set change as a result of concentration and stretching operations?
15. Give the definition of the nearest classical set to this fuzzy?
16. What set is called the set of α -level for fuzzy set?
17. Formulate a theorem on decomposition of a fuzzy set in level sets.
18. Formulate a theorem on decomposition of fuzzy sets.
19. What is called a fuzzy relation?
20. How can fuzzy relations be given?
21. What mathematical operations can be applied to fuzzy relations?
22. What fuzzy relation are called reflexive (antireflexive)?
23. What fuzzy relations belong to symmetric, antisymmetric, asymmetric?
24. What fuzzy relation is called transitive? How are the various types of transitivity of fuzzy relations interrelated?
25. What is the transitive closure of a fuzzy relation?
26. What features are taken into consideration to classify fuzzy relations?
27. Give a definition of the relation of pre-order, strict and non-strict order, equivalence, similarity, similarity, differences, preferences. What relation properties is this classification based on?

28. What is the image, mapping of a fuzzy set?
29. Formulate the generalization principle regarding the display of fuzzy sets.
30. What is the image of the fuzzy set in normal mapping?
31. What is a fuzzy mapping?
32. How can the image of the fuzzy set be determined with fuzzy mapping?
33. What is a prototype of fuzzy set in classical mapping?
34. What is a prototype of fuzzy set in fuzzy mapping?

Hands-on practice

1. The following fuzzy sets are given:

$$A = \begin{array}{c|c|c|c|c|c|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline 0{,}5 & 0{,}4 & 0{,}7 & 0{,}8 & 1 & 1 & 0{,}9 \end{array};$$

$$B = \begin{array}{c|c|c|c|c|c|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline 0 & 0{,}3 & 0{,}4 & 0{,}8 & 0{,}7 & 0{,}7 & 0{,}9 \end{array};$$

$$C:\quad \mu_C(x) = \begin{cases} -\dfrac{1}{9}(x-3)^2 + 1, \text{ if } x \in (0;6), \\ 0, \text{ if } x \notin (0;6); \end{cases}$$

$$D:\quad \mu_D(x) = \begin{cases} 0, \text{ if } x \le 0, \\ \dfrac{1}{6}x, \text{ if } x \in (0;6), \\ 1, \text{ if } x \ge 6. \end{cases}$$

Will these sets be normal? Subnormal? Identify their supports.

2. Determine the intersection and union of sets: a) *A* and *B*, b) *C* and *D* from task 1 (by three definitions).

3. Determine the complement of sets *A*, *C*.

4. Perform operations of concentration and dilatation of sets *B* and *D*.

5. Decompose the fuzzy sets *A* and *B* (from Task 1) in the level sets.

6. Find the closest to sets *A, B, C, D* of task 1 of classical sets.

7. Find the Hamming distance and Euclidean distance between sets: *a*) *A* and *B*; *b*) *C* and *D*.

8. Find the linear and quadratic indices of the fuzzy sets of *B* and *D*.

9. Calculate the linear index of fuzzy sets, which membership function $\mu(x) = 1 - (x-1)^2$, where $x \in [0; 2]$.

10. Give an example of a symmetrical and reflexive fuzzy relation.

11. Give an example of a transitive and reflexive fuzzy relation.

12. Using a matrix, define fuzzy relations: *a*) "approximately equal", *b*) "much more", in a set of numbers from 1 to 6.

13. Find max min-, min max- and max-multiplicative compositions of fuzzy relations R_1 та R_2, which are given as follows:

$$R_1 = \begin{pmatrix} 1 & 0{,}5 & 0{,}45 \\ 0{,}3 & 1 & 0{,}2 \\ 0{,}1 & 0{,}5 & 1 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 1 & 0{,}8 & 0{,}4 \\ 0{,}3 & 1 & 0{,}2 \\ 1 & 1 & 1 \end{pmatrix}.$$

14. What are the properties of each of the following fuzzy relations:

$$a)\ R = \begin{pmatrix} 1 & 0 & 0{,}2 \\ 0 & 1 & 0{,}9 \\ 0{,}2 & 0{,}9 & 1 \end{pmatrix}, \quad b)\ R = \begin{pmatrix} 1 & 0{,}5 & 0{,}45 \\ 0{,}3 & 1 & 0{,}2 \\ 1 & 1 & 0{,}4 \end{pmatrix},$$

$$c)\ R_1 = \begin{pmatrix} 0.5 & 0 & 1 \\ 0.7 & 1 & 0.2 \\ 0.20 & 0 & 0.7 \end{pmatrix}, \quad d)\ R = \begin{pmatrix} 1 & 0{,}5 & 0{,}45 \\ 0{,}3 & 1 & 0{,}2 \\ 1 & 1 & 0{,}4 \end{pmatrix}.$$

15. Let the following sets be given: $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, $Y = \{y_1, y_2, y_3\}$. Mapping $\varphi : X \to Y$ by the table, namely:

	y_1	y_2	y_3
x_1	1	0	0
x_2	0	1	0
x_3	1	0	0
x_4	0	1	0
x_5	1	0	0
x_6	0	0	1
x_7	0	0	1

Find the image φ (*A*) of the set *A* when mapping φ, if the set *A* is given as follows:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$A=$	0,5	0,4	0,7	0,8	1	1	0,9

16. Let's have the following sets: $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, $Y = \{y_1, y_2, y_3\}$. Determine the set φ (A) for the mapping φ if the set A is given as follows:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$A=$	0,5	0,4	0,7	0,8	1	1	0,9

and fuzzy mapping $\varphi: X \to Y$, set by table:

	y_1	y_2	y_3
x_1	0,7	0,5	0
x_2	0	1	0,9
x_3	0,8	0,6	0,5
x_4	0,7	0,3	0,9
x_5	1	0,7	0,6
x_6	0	0	1
x_7	0,2	0,7	1

SECTION 5

DECISION-MAKING IN FUZZY SOURCE DATA

In this section you will

- study decision-making methods in the presence of fuzzy output data and their application to solving of applied problems.

5.1 Task of achieving fuzzy goal (Bellman-Zadeh approach)

Let X be a universal set of alternatives, that is, a set of options, among which MDP chooses. A *fuzzy goal* in the set X will be referred to some its fuzzy subsets. Denote this subset as G. The fuzzy goal is described by the membership function $\mu_G : X \to [0;1]$. The higher the degree of membership of the alternative x to the fuzzy set of goals μ_G , the higher the value $\mu_G(x)$, i.e. the higher the degree to which this goal will be achieved, if you choose an alternative x as solution. Fuzzy constraints or a set of admissible alternatives are also described by fuzzy subsets of the set X. We denote them as C_1, C_2, ..., C_m. We will assume that we know the functions of the membership of these fuzzy sets.

To solve a problem means to achieve the goal and to satisfy the limitations. Moreover, in such statement it is necessary to speak not only about achievement of the goal, but about its realization at one or another degree. It is also necessary to take into account the degree of implementation of restrictions. The essence of Bellman-Zadeh approach to solving this problem is that the goal of decision-making and the set of alternatives is considered to be an equilibrium of fuzzy subsets of some universal set of alternatives. This allows to solve the problem in a relatively simple manner. In particular, in the Bellman-Zadeh approach, the requirements of the task are taken into account in the manner described below.

For example, let an alternative x provide the achievement of the goal (in other words, corresponds to the goal) with the degree $\mu_G(x)$ and satisfies the constraint (or it is admissible) with the degree $\mu_C(x)$. Under such conditions, the *fuzzy solution D,* to the task of achieving fuzzy goal is referred to the intersection of fuzzy sets of goals and constraints, i.e. $D = G \cap C$. Thus, the solution of the problem of a fuzzy defined goal also represents a fuzzy subset of the universal set of alternatives X. If the intersection of sets is determined by Rule 4.7 (see Section 4), then the membership function of the solution μ_D will have the following form:

$$\mu_D(x) = \min\{\mu_G(x), \mu_C(x)\}.$$

If there are several goals and constraints in the task, the fuzzy solution can be described by the following membership function:

$$\mu_D(x) = \min\{\mu_{G_1}(x), \dots, \mu_{G_n}(x), \mu_{C_1}(x), \dots, \mu_{C_m}(x)\}.$$

Example 5.1. Suppose we have such a universal set of alternatives: $X = \{1, 2, 3, \dots, 10\}$. The set of goals and constraints is given in this set, namely:

G – "*x* should be close to 5" (fuzzy goal),

C_1 – "*x* should not be close to 4" (first constraint),

C_2 – "*x* should be close to 6" (second constraint).

Their membership functions are given by the table

	1	2	3	4	5	6	7	8	9	10
$\mu_G(x)$	0	0,1	0,4	0,8	1	0,7	0,4	0,2	0	0
$\mu_{C_1}(x)$	0,3	0,6	0,9	1	0,8	0,7	0,5	0,3	0,2	0
$\mu_{C_2}(x)$	0,2	0,4	0,6	0,7	0,9	1	0,8	0,6	0,4	0,2

In accordance with the Bellman-Zadeh approach, the membership function of the fuzzy decision of the problem are the following values in the set *X:*

	1	2	3	4	5	6	7	8	9	10
$\mu_D(x)$	0	0,1	0,4	0,7	0,8	0,7	0,4	0,2	0	0

Obviously, such a solution is characterized by uncertainty, because we obtain more than one alternative, some fuzzy set of alternatives. If the DM is not able to process this type of decision, then we may recommend an alternative, which has the highest degree of belonging to the fuzzy solution, that is

$$\max_{x \in X} \mu_D(x) = \max_{x \in X} \min\{\mu_G(x), \mu_C(x)\}.$$

Such an alternative is called a *maximization solution*.

This is one of the most commonly used methods in the literature for choosing a single alternative.

In the example above, this solution will be number 5 since the degree of its membership to the fuzzy solution is maximal.

Example 5.2. Solve the task of achieving a fuzzy goal, when the goal and constraints are given by the following membership functions:

$$C\text{:}\quad \mu_G(x)=\begin{cases}-\frac{1}{9}(x-3)^2+1, \text{if } x\in(0;6),\\ 0, \text{if } x\notin(0;6);\end{cases}$$

$$D\text{:}\quad \mu_C(x)=\begin{cases}\frac{1}{6}x, \text{if } x\in(0;6),\\ 0, \text{if } x\notin(0;6).\end{cases}$$

Solving

To solve this problem, we will use the Bellman-Zadeh approach, i.e.

$$\mu_D(x)=\min\{\mu_G(x),\mu_C(x)\}.$$

For convenience, we will draw the graphs of the membership functions of the goal and constraints (see Figure 5.1).

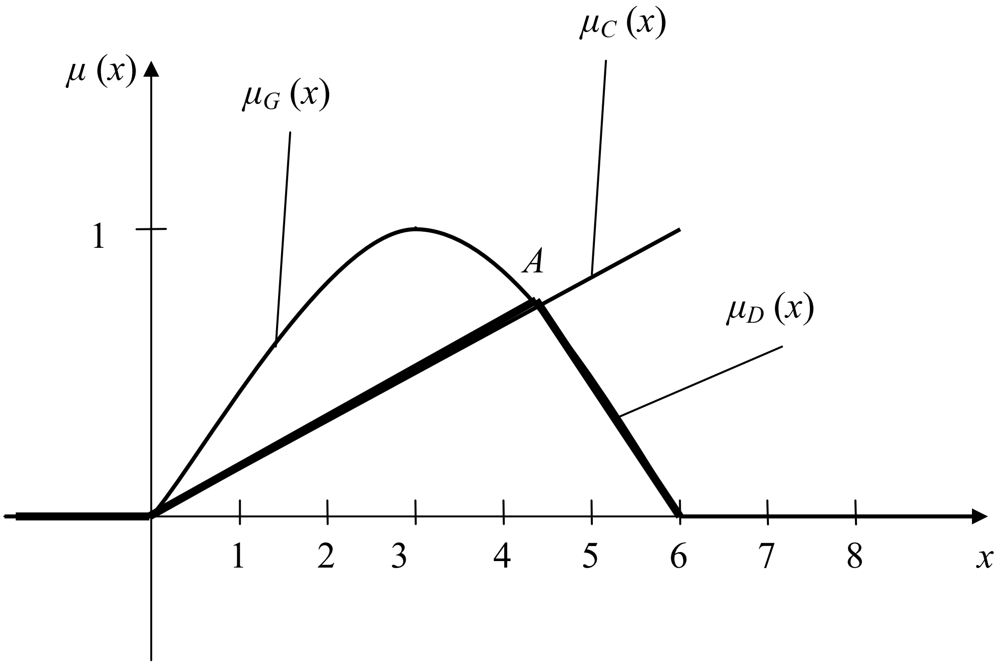


Fig. 5.1. Graphic interpretation of solving the problem of achieving a fuzzy goal

Here, a thick line shows the membership function of the fuzzy solution (decision) *D.* Let's describe it analytically. To do so we find the points of intersection of the graphs of the membership functions of the goal and its constraints by making the following equation:

$$-\frac{1}{9}(x-3)^2+1=\frac{1}{6}x.$$

Having solved it, we obtain the coordinates of two points of the cross section: $x_1 = 0$ and $x_2 = 4{,}5$. Now we can write the membership function of the solution in an analytic form, i.e.

$$\mu_D(x)=\begin{cases}\frac{1}{6}x, \text{ if } x\in(0;\ 4{,}5),\\ -\frac{1}{9}(x-3)^2+1, \text{ if } x\in(4{,}5;\ 6),\\ 0 \quad \text{in other case.}\end{cases}$$

The maximization solution is an alternative: $x_2 = 4{,}5$, and the degree of its membership to the fuzzy solution is $\mu_D(x)=0{,}75$.

The above-examined situation of decision-making was characterized by the fact that both the goal and the constraints were subsets of the same universal set. A more universal statement may be the other statement of the problem, when the fuzzy goal and constraint are subsets of the different universal sets. Let's consider it.

Suppose X is a universal set of alternatives as it was mentioned above and an image is mapped as $\varphi: X \to Y$, which value (the set Y elements) can be understood as the reaction of some system to output $x \in X$ or as some estimates of the choice of the appropriate alternatives. The map φ is considered unambiguous.

In this case, a fuzzy goal is described in the form of a fuzzy subset of the universal set of reactions (estimates) of Y, that is, the membership function $\mu_G: Y \to [0;1]$, and constraints are fuzzy subsets of the initial set X which membership functions $\mu_{C_i}: X \to [0;1]$, $i=1,2,\dots,m$.

Here, the problem is reduced to the first statement (that is, to the case when the goal serves as a fuzzy subset of the set X). Let's describe it.

We define the fuzzy set of state alternatives $\bar{\mu}_G$ that ensure the achievement of the given goal μ_G. This set is a prototype of a fuzzy set when displaying φ, i.e.

After that, the original task will be equivalent to the task of achieving a fuzzy goal from the perspective of the same fuzzy constraints.

Definition 5.1. Let G and C be fuzzy sets of goals (in the set of Y) and constraints (in the set of X). The *fuzzy solution* of the goal G goal with the constraints C is called the maximal set D, which has the following properties:

1. $D \subset C$ (the solution is an admissible alternative);

2. $\varphi(D) \subset G$ (achievement of fuzzy goal), where $\varphi(D)$ is the image of the set D with fuzzy mapping φ.

Provided the fuzzy representation of the set of alternatives in the set of reactions or estimates is given, the fuzzy solution can be found, using the definition of the prototype given in the previous Section.

Let X be a universal set of alternatives, Y is a universal set of estimates, and also gives a fuzzy mapping of X to Y, which membership function is $\mu_\varphi : X \times Y \to [0;1]$. Each alternative of this mapping matches its fuzzy rating. Fuzzy constraints are described by membership function $\mu_C(x)$. By Theorem 4.6, the prototype of the set D is defined as follows:

$$N = \{(x, y) \mid (x, y) \in X \times Y,\ \mu_\varphi(x, y) > \mu_G(y)\},$$

$$N_x = \{y \mid y \in Y,\ (x, y) \in N\},$$

$$X^0 = \{x \mid x \in X,\ N_x \neq \varnothing\},$$

$$\mu_{\tilde{D}}(x) = \begin{cases} \inf\limits_{y \in N_x} \mu_G(y), & x \in X, \\ 1, & x \in X \setminus X^0. \end{cases}$$

Now the fuzzy solution is described by the following membership function:

$$\mu_D(x) = \min\{\mu_{\tilde{D}}(x), \mu_C(x)\},$$

or

$$\mu_{\tilde{D}}(x) = \begin{cases} \min\left\{\mu_C(x), \inf\limits_{y \in N_x} \mu_G(x)\right\}, & x \in X,^0 \\ \mu_C(x), & x \in X \backslash X^0. \end{cases}$$

If it is necessary to choose a concrete alternative, then for solution of the problem it is possible, for example, to choose the one which degree of membership μ_D is maximal to the fuzzy solution, that is, the alternative that implements the value $\max\limits_{x \in X} \mu_D(x)$. However, this choice cannot be considered sufficiently substantiated. There are also other ways of defining a single alternative

Thus, the Bellman-Zadeh approach is based on the possibility of symmetric description of the sets of goals and constraints in the form of fuzzy subsets of one and the same universal set. This allows you to solve the problem in a rather simple way. At the same time, not every decision task can be formulated in this way.

Remark: Occasionally, the importance of goal and constraints is taken into account by weighting factors. Then the solution to the problem is described as follows:

$$\mu_D(x) = \min\{\lambda_1\mu_{G_1}(x),\ldots,\lambda_n\mu_{G_n}(x),\nu_1\mu_{C_1}(x),\ldots,\nu_m\mu_{C_m}(x)\},$$

where $\lambda_1,\ldots,\lambda_n,\nu_1,\ldots,\nu_m$ weighted coefficients of objective functions and constraints, respectively.

This approach cannot be considered sufficiently substantiated.

5.2 Fuzzy mathematical programming problems and their classification

The standard problem of mathematical programming is usually a search for the maximum (or minimum) of a given function in a given set of admissible alternatives, which is described by the system of inequalities. Example,

$$f(x) \to \max,$$

$$\varphi_i(x) \le 0, i = 1, \ldots, m,$$

$$x \in X,$$

where X is a set of alternatives given, $f : X \to R^1$ and $\varphi_i : X \to R^1$, $i = 1,\ldots,m$ are given functions.

At the same time, when simulating real problems, the researcher can often have only fuzzy descriptions of the functions f and φ or their parameters, the set of alternatives X cannot be described clearly. Such a representation of the decision-making situation can, for example, reflect the inadequacy of the available information or to be a form of an approximate description sufficient for solving the problem.

Moreover, in some cases the precisely defined set of constraints (admissible alternatives) can only be approximate to the real situation in the sense that in the original problem alternatives beyond the set of constraints may not be inadmissible, but are only to some extent less desirable for DM. For example, let's recall a situation where the set of acceptable alternatives is a combination of all sorts of ways of distributing resources that the DM is going to invest in this operation. In this case, it is impractical to enter a clear limit of the set of permissible alternatives (distributions) in advance, since it may happen that the allocation of resources which is under this is threshold will give an effect outweighting "less" its desirability for the DM. Thus, a fuzzy description may prove to be more adequate to reality than arbitrarily accepted clear in some sense limitations.

Forms of fuzzy description of information may be different. Hence, this causes differences in mathematical formulations of problems of *fuzzy mathematical programming* (FMP). Some of them have been grouped in 5 types of tasks described below [27].

Task 1. Maximization of a given classical function on a fuzzy set of alternatives. That is, we have the following task:

$$f(x) \to \max,$$

$$x \in X,$$

where $f : X \to R^1$, $\mu_C : X \to [0;1]$.

Let's describe approaches to solving this problem.

1. *Reducing the task to a fuzzy goal.*

For this purpose, the objective function is standardized as follows:

$$\bar{f}(x) = \frac{f(x)}{\sup\limits_{x \in \operatorname{supp} \mu_C} f(x)} \to \max.$$

The resulting function $\bar{f}(x)$ is considered to be a function of belonging to the fuzzy goal. Therefore, the value $\bar{f}(x)$ will be the degree to which the goal is achieved when choosing an alternative $x \in X$. This allows to apply the Bellman-Zadeh approach to solving this problem directly. The choice of an alternative is considered rational when the alternative has maximum degree of membership of the fuzzy solution, that is, implements the following value:

$$\max_{x \in X} \min\{\mu_C(x), \bar{f}(x)\}.$$

2. *Reducing the task to the problem of multi-criteria optimization.*

In this approach, the fact that it is necessary to achieve the maximum value of the function and the maximal membership of the solution of the problem to the set of admissible alternatives is taken into account. Thus, we formulate such a multicriteria problem:

$$
\begin{aligned}
&f(x) \to \max, \\
&\mu_C(x) \to \max, \\
&x \in X.
\end{aligned}
$$

This approach will be further discussed in more detail below.

Task II. This is a fuzzy version of the standard problem of mathematical programming.

It can be obtained if you "mitigate" constraints that is, assume the possibility of some of their violation in the standard problem of mathematical programming, namely:

$$
\begin{aligned}
&f(x) \to \max, \\
&\varphi(x) \tilde{\le} 0, \\
&x \in X.
\end{aligned}
$$

Besides, instead of maximizing the function *f (x),* one can strive to achieve a fixed value of this function. Moreover, a different deviation of *f (x)* of this value should be assigned different degrees of admissibility, For example, the larger the deviation, the lower the degree of its admissibility. In this case, the fuzzy problem can be written as following:

$$
\begin{aligned}
&f(x) \ge z_\circ, \\
&\varphi(x) \tilde{\le} 0, \\
&x \in X,
\end{aligned}
$$

where the symbol $^{\sim}$ means the fuzziness of the corresponding inequalities.

We describe one of the ways of formalizing such tasks.

Assume that z_0 is a given value of the objective function $f(x)$, the achievement of which is considered sufficient for the purpose of decision-making, and there are (given to DM) two limit levels a and $b,$ and inequality: $f(x) < z_\circ - a$ signifies a strong violation of the condition: $f(x) \ge z_\circ$, a $\varphi(x) > b$ is a strong violation conditions: $\varphi(x) \le 0$. Then you can write sets of goals and constraints using the following membership functions:

$$
\mu_G(x) = \begin{cases} 0, & f(x) \le z_\circ - a, \\ \mu(x, a), & z_\circ - a < f(x) < z_\circ, \\ 1, & f(x) \ge z_\circ, \end{cases}
$$

$$\mu_C(x)=\begin{cases}0, & \varphi(x)\geq b,\\ \nu(x,b), & 0<\varphi(x)<b,\\ 1, & \varphi(x)\leq 0,\end{cases}$$

where $\mu : X \to [0;1]$ and $\nu : X \to [0;1]$ is some of the functions that describe the extent to which the inequalities are implemented from the point of view of DM and taking into account the specific task of decision-making.

Thus, the original problem will be formulated in the form of the task of achieving a fuzzy goal, to which the Bellman-Zadeh approach can be applied, or can be reduced to the multi-criterion optimization problem of this kind:

$$\begin{aligned}&\mu_G(x)\to \max,\\ &\mu_C(x)\to \max,\\ &x\in X.\end{aligned}$$

More detailed description of the methods for solving this problem will be discussed below (Section 5.3).

Task III. It is given the vaguely described function that needs to be "maximized", that is, a mapping $\mu_\varphi : X\times R^1 \to [0;1]$, where X is a universal set of alternatives, R^1 is a numerical axis. In this case, the function $\mu_\varphi(x_0, r)$ for each fixed $x_0 \in X$ is a fuzzy evaluation of the result of the choice of alternative x_0 (fuzzy *assessment* of the alternative x_0) or a fuzzy reaction of the system to control x_0. The fuzzy set of admissible alternatives is also given as $\mu_C : X \to [0;1]$.

Many classes of tasks of fuzzy mathematical programming are described in such a statement. Methods of their solution are discussed in the monograph [25].

Task IV. It is specified the usual objective function $f : X \to R^1$ and constraint system: $\varphi_i(x)\leq b_i$, i = 1, ..., m. Moreover, the parameters in the description of functions $\varphi_i(x)$ are given vaguely in the form of fuzzy sets, in particular.

For example, in the linear case $\left(X = R^n\right)$, the functions $\varphi_i(x)$ have the following form:

$$\varphi_i(x)=\sum_{i=1}^{n} a_{ij}x_j, \quad i = 1, \dots , m,$$

and each of the parameters a_{ij} and b are described by the corresponding fuzzy set $\mu_{ij}(a_{ij})$, $\nu_i(b_i)$.

Several ways of solving such problems are developed.

One of them is the method of modal values. Its essence is that the fuzzy parameter is replaced by its modal value, and then the scalar problem is solved. The degree of membership of the resulting solution is calculated as a minimum among the degrees of membership of the parameters of modal values. However, this method can be used when the membership function of the parameters is unimodal, that is, each of them reaches its maximum value only at one point. If this requirement is not met, then the question of which of the values *f* the parameters, having the highest degree of membership to be chosen, remains open.

Another method of solving is to construct the original problem bringing to Task I type.

There are also methods based on the construction of the original problem to the problem of multi-criteria optimization [6].

Task V. In its condition, the parameters of constraints functions and parameters of the objective function constraints are not clearly described.

One of the approaches to solving such tasks is its reduction to the Task III type.

5.3 Problems of mathematical programming with fuzzy constraints

Let a function $\varphi : X \to R^1$ is set, the values of which evaluate the results of the choice of alternatives, and a fuzzy subset of acceptable alternatives of alternatives of X $\mu_C : X \to [0;1]$ are given in the universal set. It is worthwhile to "maximize" a function φ in a certain sense in a fuzzy set μ_C , that is

$$\varphi(x) \to \max,$$
$$x \in \widetilde{C}.$$

This means that by "maximizing" one can understand the choice of a fuzzy subset (fuzzy solution), which corresponds better in some sense to the fuzzy value of the function φ. It is clear the presentation of the solving in this form is expedient only when it is meaningfully perceived by the DM.

If DM does not perceive a fuzzy description of the problem, then by "maximizing" the function φ one should understand the rational choice of a concrete alternative or a set of alternatives.

Herein, rationality means that when choosing a particular alternative, DM should proceed from the need for a compromise between the desire to get the highest value of function φ and the desire to give preference to an acceptable alternative that has the greatest degree of belonging to the set of permissible alternatives.

We consider two approaches to the solution of this problems. Their justification is given in the monograph [27].

5.3.1 Solution 1 based on the level sets of fuzzy set of constraints

This approach is that the initial problem of fuzzy mathematical programming is formulated in the form of a set of classical problems of maximizing a function φ in all possible level sets of a set of admissible alternatives. If the alternative $x_0 \in X$ is a solution of the problem: $\varphi(x) \to \max$, in the λ-level set, then it is natural to assume that its degree of belonging to the fuzzy set of solutions is at least λ.

Thus, taking all possible values λ, we obtain the fuzzy solution membership function.

Let`s describe this approach in more detail.

Let`s denote the λ- level set of the fuzzy set of admissible alternatives μ_C by C_λ, that is,

$$C_\lambda = \{x \mid x \in X, \mu_C(x) \geq \lambda\}.$$

For all numbers $\lambda \geq 0$, provided that $C_\lambda \neq \varnothing$, we introduce such a set:

$$N(\lambda) = \left\{ x \,\middle|\, x \in X, \varphi(x) = \sup_{x' \in C_\lambda} \varphi(x') \right\}.$$

This is the set of solutions of the ordinary problem of maximizing the function φ in the set of alternatives, the degree of membership of which to the set of admissible alternatives of the initial FMP problem is not less than λ.

To construct the fuzzy solution membership function, it is necessary to assign to each alternative $x \in X$ the degree of belonging to this set. We do this in the following way: the degree of belonging to the alternative x_0 of an fuzzy set of solutions is the maximum (more precisely, the upper bound) of the numbers λ for which the corresponding set $N(\lambda)$ contains an alternative x_0.

Definition 5.2. The solution of the FMP problem will be referred to as a fuzzy subset, which is described by such membership function:

$$\mu^1(x) = \sup_{\lambda : x \in N(\lambda)} \lambda. \qquad (5.1)$$

Let's call it solution of type 1.

Statement 5.1 If $x \in \operatorname{supp} \mu^1(x)$, then $\mu^1(x) = \mu_C(x)$

Proof

a) When $x \in \operatorname{supp} \mu^1(x)$ and $\mu^1(x) > \mu_C$, then $\sup\limits_{\lambda : x \in N(\lambda)} \mu^1(x) > \mu_C(x)$. This means that there is number $\tilde{\lambda}$ that satisfies the conditions: $\tilde{\lambda} > \mu_C(x)$, and $x \in N(\tilde{\lambda})$. Then, according to the definition of a fuzzy solution $x \in C_{\tilde{\lambda}}$, then $\mu_C(x) \geq \tilde{\lambda}$, that is, the set: $\tilde{\lambda} > \mu_C(x)$, is impossible.

b) If $x \in \operatorname{supp} \mu^1(x)$ and $\mu^1(x) < \mu_C(x)$, that is:

$$\sup_{\lambda : x \in N(\lambda)} \mu^1(x) < \mu_C(x) = \nu, \tag{5.2}$$

then, for any number λ which satisfies the condition $x \in N(\lambda)$, the inclusions $x \in C_\nu \subset C_\lambda$ are fulfilled, in addition, $x \notin N(\lambda)$, since otherwise it follows from (5.2) that $\nu < \nu$, hence

$$\varphi(x) < \sup_{x' \in C_\nu} \varphi(x') \leq \sup_{x' \in C_\lambda} \varphi(x') = \varphi(x).$$

Thus, the statement has been proved.

Taking into account Statement 5.1 and Definition 5.2, the membership function of the solution of type 1 can be written in the following form:

$$\mu^1(x) = \begin{cases} \mu_C(x), \text{ if } x \in \bigcup\limits_{\lambda > 0} N(\lambda), \\ 0 \text{ in other case,} \end{cases} \tag{5.3}$$

thus,

$$\operatorname{supp} \mu^1(x) = \bigcup_{\lambda > 0} N(\lambda).$$

We will say that solution 1 exists, if $\mu^1(x) \neq 0$ in the set *X*, that is, if only there is a number that $\lambda > 0$ for which $N(\lambda) \neq \varnothing$.

A fuzzy solution corresponds to a *fuzzy* "maximum" value $\mu_\varphi(r)$ of the function $\varphi(x)$. It is an image of a fuzzy set $\mu^1(x)$ under the mapping φ and in accordance with Definition 4.36, it looks like that:

$$\mu_\varphi(r) = \sup_{x \in \varphi^{-1}(r)} \mu^1(x) = \sup_{x \in \varphi^{-1}(r)} \sup_{x \in N(\lambda)} \lambda, \tag{5.4}$$

where

$$\varphi^{-1}(r) = \{x | x \in X, \varphi(x) = r\}.$$

If there is no solution of type 1 in the FMP problem, then we can use the *ε-optimal* fuzzy solution, which for a given number $\varepsilon > 0$, can be defined as follows:

$$\mu_\varepsilon^1(x) = \sup_{\lambda : x \in N(\varepsilon,\lambda)} \lambda, \tag{5.5}$$

where
$$N(\varepsilon,\lambda) = \left\{ x \,\middle|\, x \in X, \varphi(x) \ge \sup_{x' \in C_\lambda} \varphi(x') - \varepsilon \right\}, \tag{5.6}$$

and a fuzzy value of the function φ corresponding to it is described by the following membership function:

$$\mu_\varphi^\varepsilon(x) = \sup_{x \in \varphi^{-1}(r)} \mu_\varepsilon^1(x). \tag{5.7}$$

The boundary μ_φ^ε, when $\varepsilon \to 0$, can be considered as the upper fuzzy boundary of the function φ on the fuzzy set μ_C.

The concept of ε-optimal solution can be useful not only when $\mu^1(x) = 0$ for all alternatives $x \in X$, but also when $N(\lambda) = \varnothing$ for certain values of λ from the interval [0; 1].

Consider the properties of solution 1.

1. For any number of r_0, provided by the condition that $\mu_\varphi(r_0) > 0$, there is an alternative: $\tilde{x} \in X$, for which $\varphi(\tilde{x}) = r_0$ and $\tilde{x} \in N(\lambda)$ for a value $\lambda > 0$, that is,

$$r_0 \in \operatorname{supp} \mu_\varphi \Rightarrow \left[\bigcup_{\lambda > 0} N(\lambda) \right] \cap \varphi^{-1}(r_0) \ne \varnothing. \tag{5.8}$$

Proof

According to the definition $\mu_\varphi(\cdot)$ and the set $\operatorname{supp} \mu_\varphi$ on the left side of expression (5.8), we obtain the following result:

$$\sup_{x \in \varphi^{-1}(r_0)} \sup_{\lambda : x \in N(\lambda)} \lambda > 0,$$

that is, there is an alternative $x' \in \varphi^{-1}(r_0)$, for which there will be fair inequality: $\sup_{\lambda : x' \in N(\lambda)} \lambda > 0$ and this, in its turn, means that there is a number $\lambda > 0$ for which the alternative is $x' \in N(\lambda)$. From here we obtain the inclusion $x' \in \varphi^{-1}(r_0) \cap N(\lambda)$, which proves the truth of the expression (5.8).

2. If $r_0 \in \operatorname{supp} \mu_\varphi$, then $\sup\limits_{x \in \varphi^{-1}(r_\circ)} \mu^1(x) = \sup\limits_{x \in \varphi^{-1}(r_\circ)} \mu_C(x)$.

Proof

Taking into account the definition of a fuzzy solution, membership functions $\mu^1(x)$ and $\mu_C(x)$ related by inequality: $\mu^1(x) \le \mu_C(x)$ for each alternative $x \in X$, so this inequality will also be true:

$$\sup_{x \in \varphi^{-1}(r_\circ)} \mu^1(x) \le \sup_{x \in \varphi^{-1}(r_\circ)} \mu_C(x).$$

Suppose that for a certain number $r_0 \in \operatorname{supp} \mu_\varphi$ it holds as strict. Then with respect to some alternative $x_0 \in \varphi^{-1}(r_0)$, the inequality:

$$\mu^1(x) < \mu_C(x_0) \tag{*}$$

is performed for all alternatives $x \in \varphi^{-1}(r_0)$.

Furthermore, as $r_0 \in \operatorname{supp} \mu_\varphi$, according to the property 1, there is an alternative $x' \in X$ and a number: $\lambda > 0$, for which the valid statement is true: $x' \in \varphi^{-1}(r_\circ) \bigcap N(\lambda)$. Since $x' \in N(\lambda)$ and $\lambda > 0$, then taking into account the definition of the solution, $\mu^1(x') = \mu_C(x') \ge \lambda$ and thus, according to the inequality (*) $\mu_C(x_0) > \lambda$, that is $x_0 \in C_\lambda$. And from the perspective that, $x' \in N(\lambda)$ and $x \in \varphi^{-1}(r_0)$, that is $\varphi(x') = r_0$, we have such equality:

$$\varphi(x') = \sup_{x \in C_\lambda} \varphi(x) = r_0 = \varphi(x_0),$$

i.e. $x_0 \in \varphi^{-1}(r_0)$. Hence $x_0 \in N(\lambda)$, and in accordance with Statement 5.1, $\mu^1(x_0) = \mu_C(x_0)$, which contradicts to the inequality (*).

Property 2 has been proved.

3. The function $\mu_\varphi(r)$ monotonically decreases in the set $\operatorname{supp} \mu_\varphi$.

Proof

It is enough to show that $\mu_\varphi(r_1) \le \mu_\varphi(r_2)$ for any values $r_1, r_2 \in \operatorname{supp} \mu_\varphi$, satisfying the inequality: $r_1 > r_2$.

Assume the opposite, that is, for some numbers $r_1 > r_2$ from the set $\operatorname{supp} \mu_\varphi$ the inequality $\mu_\varphi(r_1) > \mu_\varphi(r_2)$ is performed. Then, according to the Property 2

$$\sup_{x\in\varphi^{-1}(r_1)} \mu_C(x) > \sup_{x\in\varphi^{-1}(r_2)} \mu_C(x),$$

that is, there is an alternative $x_1 \in \varphi^{-1}(r_1)$ that satisfies such inequality:

$$\mu_C(x_1) > \mu_C(x), \qquad (**)$$

for all the alternatives $x \in X$.

Furtherore, since $r_2 \in$ supp μ_φ ,then taking into account the given Property 1, there is an alternative $x_2 \in \varphi^{-1}(r_2)$and the number $\lambda > 0$ corresponding to the following condition: $x_2 \in N(\lambda)$, that is $r_2 = \varphi(x_2) = \sup_{x\in C_\lambda} \varphi(x)$

Besides, from here and the inequality (**) it follows that $x_1 \in C_\lambda$, and that is why

$$r_2 = \varphi(x_2) = \sup_{x\in C\lambda} \varphi(x) \geq \varphi(x_1) = r_1,$$

that is $r_2 \geq r_1$, and this contradicts the assumption that $r_1 > r_2$.

Property 3 has been proved.

So, the function $\mu_\varphi(r)$ is described in such a way that its value for a particular number $r \in R^1$ is the maximum degree of belonging of the alternative *x* to the set $\mu_C(x)$, within the framework of which the function $\varphi(x)$acquires value *r*.

As it has been proved in Property 3, function $\mu_\varphi(r)$ decreases in the set supp $\mu_\varphi$monotonically. This means, in particular, that within the set *X* there is no alternatives for which the inequalities $\mu_C(x) > \mu_\varphi(x) > 0$ and $\varphi(x) > r$ would be fulfilled at the same time, that is, there is no such element $x \in X$ that should have been greater than $\mu_\varphi(r)$ the degree of membership $\mu_C(x)$ and would provide more *r* value than the maximized function.

If a fuzzy solution is unacceptable for DM and it is necessary to choose a specific alternative $x \in X$, then this choice should rely not only on the degree of belonging of this alternative to the fuzzy set $\mu_C(x)$, but also on the corresponding value of the function $\varphi(x)$. As follows from Property 3, the larger the value of r_0, the smaller the value $\mu_C(x)$ of the degree of belonging of that alternative *x* that achieves this value. Under these conditions, the DM must first turn to the fuzzy maximum value $\mu_\varphi(r)$ of the function $\varphi(x)$ and select a pair $(r_0, \mu_\varphi(r_0))$ that corresponds to its desire to obtain as large a value r_0, and at the same time the highest degree of its membership in the set $\mu_\varphi(r)$. After having chosen such a pair, it is appropriate to stop on such an alternative $x_0 \in \varphi^{-1}(r_0)$ that has the greatest degree of belonging to the set $\mu_C(x)$ (or alternative, which in a sense is close to *x*).

There are two main disadvantages of this approach.

Firstly, this solution does not clearly take into account the need for a compromise between the values of the maximized function and the values of the degree of belonging of the alternative to the set of admissible solutions.

Secondly, it is complicated for calculation.

If the membership function is continuous, then the application of this approach requires consideration of an infinite number of tasks, since the number of sets of the level will be infinite. However, in the case of its practical use, it will suffice to consider a finite set of tasks for sets of levels defined by experts or DM.

Let`s consider the application of the described approach for the linear problem of FMP.

Example 5.3. Solve this task of fuzzy mathematical programming:

$$\begin{aligned}
&f(x_1, x_2) = 2x_1 + 5x_2 \to \max,\\
&2x_1 + 3x_2 \geq 6,\\
&x_1 + 6x_2 \leq 18,\\
&2x_1 + 3x_2 \leq\approx 12,\\
&x_1, x_2 \geq 0,
\end{aligned}$$

when fuzzy constraints are described by such a membership function:

$$\mu(x) = \begin{cases} 0, \text{ if } 2x_1 + 3x_2 \geq 14, \\ 0{,}5, \text{ if } 13 \leq 2x_1 + 3x_2 \leq 14, \\ 0{,}7, \text{ if } 12 \leq 2x_1 + 3x_2 \leq 13, \\ 1, \text{ if } 2x_1 + 3x_2 \leq 12. \end{cases}$$

Solving

Let`s apply a method of expansion on level sets. Taking into account the form of the membership function of constraints, it is necessary to solve tasks on such sets of level: $\lambda_1 = 1$; $\lambda_2 = 0{,}7$; and $\lambda_3 = 0{,}5$.

At the level: $\lambda_1 = 1$, the task acquires the following form:

$$
\begin{aligned}
&f(x_1, x_2) = 2x_1 + 5x_2 \to \max \\
&2x_1 + 3x_2 \ge 6, \\
&x_1 + 6x_2 \le 18, \\
&2x_1 + 3x_2 \le 12, \\
&x_1, x_2 \ge 0.
\end{aligned}
$$

When λ_2 = 0,7, then we have the following task:

$$
\begin{aligned}
&f(x_1, x_2) = 2x_1 + 5x_2 \to \max \\
&2x_1 + 3x_2 \ge 6, \\
&x_1 + 6x_2 \le 18, \\
&2x_1 + 3x_2 \le 13, \\
&x_1, x_2 \ge 0.
\end{aligned}
$$

If level λ_3 = 0,5, the task will be written as follows:

$$
\begin{aligned}
&f(x_1, x_2) = 2x_1 + 5x_2 \to \max \\
&2x_1 + 3x_2 \ge 6, \\
&x_1 + 6x_2 \le 18, \\
&2x_1 + 3x_2 \le 14, \\
&x_1, x_2 \ge 0.
\end{aligned}
$$

The formulated tasks are linear programming problems, and for their solution, the simplex method can be used, for example, or since each of them has only two variables, it is possible to solve them graphically.

With this purpose, we represent the sets of fuzzy level set of admissible alternatives on the coordinate plane (see Figure 5.2).

Here, the *ABCDE* polygon corresponds to the level of the set: λ_1 = 1; $A_1 BCDE_1$ to level of the set: λ_2 = 0,7; $A_2 BCDE_2$ -level of the set: λ_3 = 0,5.

The solution of the problem at the set of level 1 will be the point A. We find its coordinates from the following system of equations:

$$
\begin{cases}
x_1 + 6x_2 = 18, \\
2x_1 + 3x_2 = 12.
\end{cases}
$$

We get the following result: $x_1 = 2, x_2 = 2\frac{2}{3}.$

The value of the objective function at this point is $f(A) = 17\frac{1}{3}$.

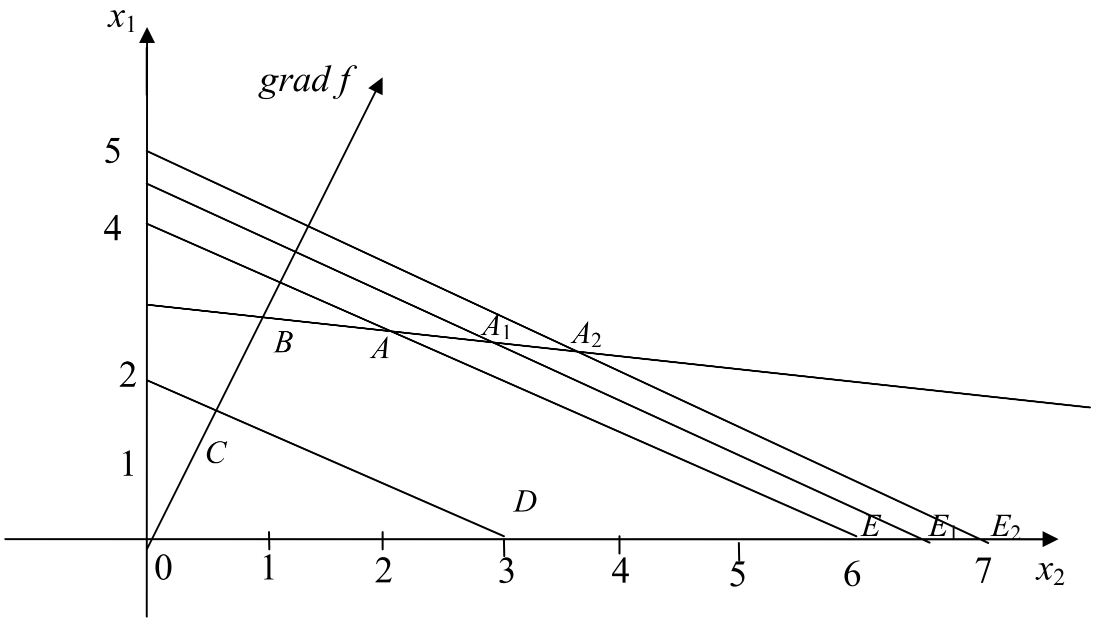


Fig. 5.2. Graphic interpretation solving of the fuzzy mathematical programming problem (Example 5.2)

Similarly, we solve the tasks at sets of 0.7 and 0.5 level. As a result, we obtain the point A_1, which has the following coordinates: $x_1 = 2\frac{2}{3}, x_2 = 2\frac{5}{9}$, and the point A_2 (its coordinates: $x_1 = 3\frac{1}{3},\ x_2 = 2\frac{4}{9}$). The values of the objective function corresponding to these points are: $f(A_1) = 18\frac{1}{9},\quad f(A_2) = 18\frac{2}{3}$.

Let`s describe the fuzzy solution of the problem by gathering the received results into a table.

x_1	x_2	f	λ
2	$2\frac{2}{3}$	$17\frac{1}{3}$	1
$2\frac{2}{3}$	$2\frac{5}{9}$	$18\frac{1}{9}$	0,7
$3\frac{1}{3}$	$2\frac{4}{9}$	$18\frac{2}{3}$	0,5

We would like to remind that when membership functions are continuous, then it is necessary to solve tasks on an infinite number of sets of levels. But in practice it is enough to define several levels for consideration.

5.3.2 Solution 2 based on finding set of effective alternatives. Equivalence of solutions of both types

This approach is characterized by the fact that, from the very beginning, the desire of DM to choose an alternative to obtain as much as possible maximized function and the membership function of the fuzzy set of admissible alternatives is explicitly taken into account.

With the same purpose, only the alternatives which are called efficient after Pareto in multicriteria optimization tasks are to be used in solving this task.

We would like to remind that an alternative $x_0 \in X$ is called effective in two functions $\varphi(x)$ and $\mu_C(x)$ when for any other alternative $x' \in X$ from the inequalities: $\varphi(x') \geq \varphi(x_0)$ and $\mu_C(x') \geq \mu_C(x_0)$, follows from the validity of the following equations: $\varphi(x') = \varphi(x_0)$ and $\mu_C(x') = \mu_C(x_0)$.

In other words, if x_0 is an effective alternative for functions $\varphi(x)$ and $\mu_C(x)$ in the set X, then by choosing any alternative, it is not possible to increase, in comparison with $\varphi(x_0)$ and $\mu_C(x_0)$, the value of one function without diminishing the value of another.

In the task of making decisions when there are several criteria, the set of effective alternatives serves as a set of proposed options for rational choice being implemented by DM.

Consequently, let P be the set of all effective alternatives for functions $\varphi(x)$ and $\mu_C(x)$ which are considered in the task of fuzzy mathematical programming.

Definition 5.3. The solution of the DM problem is called the fuzzy set whose membership function is

$$\mu^2(x) = \begin{cases} \mu_C(x), & \text{if } x \in P, \\ 0, & \text{in other case.} \end{cases} \tag{5.9}$$

We will call it as a solution of type 2.

In this definition, there is a clear assumption that while making a decision, a DM should use only alternatives to the universal set X, that are at the same time unbending values of the functions $\varphi(x)$ and $\mu_C(x)$.

Corresponding to solution 2, the fuzzy value of the function $\varphi(x)$ is written as follows:

$$\mu_\varphi^2(r) = \sup_{x\in\varphi^{-1}(r)} \mu^2(x), \qquad r \in R^1. \tag{5.10}$$

Let`s establish the relationship between the solutions of both types.

Theorem 5.1 [27]. If the set *X* is compact, the function $\varphi(x)$ is continuous, and the function $\mu_C(x)$ is semi-continuous on the top of the set *X*, then, for each value $r \in R^1$ the following equality holds:

$$\mu_\varphi^1(r) = \mu_\varphi^2(r). \tag{5.11}$$

Proof

Note, at first, that in this theorem, the set $\varphi^{-1}(r)$ is closed in *X* relatively to each number $r \in R^1$ and thus, for $\forall\, r \in R^1$ an alternative $x_0 \in \varphi^{-1}(r)$ can be found which satisfies the following equality:

$$\sup_{x\in\varphi^{-1}(r)} \mu^1(x) = \mu^1(x_0). \tag{5.12}$$

Assume that there is a number $r_0 \in R^1$ for which either $\mu_\varphi^1(r_0) > \mu_\varphi^2(r_0)$ or

$$\sup_{x\in\varphi^{-1}(r_0)} \mu^1(x) > \sup_{x\in\varphi^{-1}(r_0)} \mu^2(x). \tag{5.13}$$

As a consequence of equality (5.12) and in the conditions of the theorem, the expression (5.13) takes on the following form:

$$\mu_C(x_0) = \max_{x\in\varphi^{-1}(r_0)} \mu_C(x) = \sup_{x\in\varphi^{-1}(r_0)} \mu^1(x) > \sup \mu^2(x). \tag{5.14}$$

Therefore,

$$\mu_C(x_0) > \mu^2(x), \ \ \forall x \in \varphi^{-1}(r_0). \tag{5.15}$$

The following situations are possible:

a) If $\mu^2(x) > 0$ for some element $x \in \varphi^{-1}(r_0)$, then $\mu^2(x_0) = \mu_C(x)$, and $\varphi(x_0) = \varphi(x) = r_0$, $\mu_C(x_0) > \mu_C(x)$. But this contradicts the effectiveness of the alternative *x* for the functions $\varphi(x)\, \mu_C(x)$ and $\mu_C(x)$.

b) If $\mu^2(x)=0$, $\forall x \in \varphi^{-1}(r_0)$, then no alternative is effective, that is, for any alternative $x \in \varphi^{-1}(r_0)$, there is another alternative $x' \in X$ that meets the following conditions:

$$\varphi(x) \geq \varphi(x') = r_0, \ \ \mu_C(x) > \mu_C(x') \tag{5.16}$$

or

$$\varphi(x) > \varphi(x') = r_0, \ \ \mu_C(x) \geq \mu_C(x'). \tag{5.17}$$

But if $x' \in \varphi^{-1}(r_0) \bigcap N(\lambda)$ for some number $\lambda > 0$, then on the basis of conditions (5.17) we conclude that $x \in C_\lambda$. Thus

$$\varphi(x) \leq r_0 = \varphi(x'),$$

and this contradicts the inequalities (5.17).

With respect to the conditions (5.16), they do not have the place for an alternative $x' = x_0$ because $x \in \varphi^{-1}(r_0)$ and $\mu_C(x_0) = \max_{x \in \varphi^{-1}(r_0)} \mu_C(x)$.

From here $\mu_\varphi^1(r) = \mu_\varphi^2(r), \ \ \forall r \in R^1$. Hence, the theorem has been proved.

Given the definition of 5.3, the implementation of the solution of type 2 is reduced to finding the set of effective alternatives of functions $\varphi(x)$ and $\mu_C(x)$. However, this set includes, in the general case, an infinite number of elements, and its construction is a rather complex task.

However, in order to obtain such a solution in a particular task, it is sufficient that a finite number of effective alternatives, uniformly selected from the set *P*, have been determined.

To find them, you can use the following property (see Section 3, Theorem 3.1):

If there are numbers ν_1, ν_2 ($\nu_1 > 0$, $\nu_2 > 0$, $\nu_1 + \nu_2 = 1$), for which the alternative x_0 ensures the achievement of the maximum of the function $F(x) = \nu_1 \varphi(x) + \nu_2 \mu_C(x)$ in the set *X*, then this alternative is effective for the objective functions $\varphi(x)$ and $\mu_C(x)$.

Thus, by providing various positive values of the weighted functional coefficients $\varphi(x)$ and $\mu_C(x)$, and maximizing the corresponding functions $F(x)$, it is possible to determine any necessary number of effective alternatives.

The alternatives, obtained along with the relative values of functions $\varphi(x)$and $\mu_C(x)$, are transferred to the DM who makes the final choice from their subjective representations (or using information not taken into account in this mathematical model) about the relative importance of the values of functions $\varphi(x)$and $\mu_C(x)$.

5.4 Decision-making when fuzzy relation of preferences is given in a set of alternatives

Investigating a real situation or process in order to make a rational decision, it is natural first to identify the set of all permissible solutions or alternatives.

Depending on the quality of information we have, this set can be described with varying degrees of clarity. Let, for example, have some universal set of alternatives X and an fuzzy description of its subset of admissible alternatives $\mu_C(x)$. The value of the function μ_C describes the extent to which the appropriate alternatives in the given task are admissible.

If in addition to this function, there is no other information about the investigated alternative, then it is rational to choose some alternative from the following set:

$$X^D = \left\{ x \middle| x \in X,\ \mu_C(x) = \sup_{y \in X} \mu_C(y) \right\}.$$

In other words, it is advisable to choose an arbitrary alternative from those with the highest degree of admissibility, since there is no reason to give preference to others. When introduced into the model of additional information rational, there may be the choice of alternatives from any subset of the set X^D or any alternatives that do not belong to this set. It is possible that this information may serve as the basis for identifying a single, the best of all alternative.

Information about a real situation or process, guided by the preferences of one alternative over another, can be expressed in different ways. In the previous sections we have already considered instances when it was presented in the form of utility functions or described by numerical inequalities, but this method of description is not always possible. More universal can be considered to be the presentation of information in the form of preferences to the set of alternatives, in particular in the form of binary relations (this case was discussed in Section 2), but they can not always be clearly defined. In other words, sometimes a more precise model of the situation will be a description of the preferences in the form of fuzzy relations, that is, when they appear only to some extent. The ways of making decisions under such conditions will be discussed below.

5.4.1 Fuzzy relation of preferences and their properties

Definition 5.3. Let X be a given set of alternatives. The *fuzzy relation of non-strict preference (*FRP) in the set X will be referred to as any reflexive relation given in it.

This relation is described by the membership function: $\mu_R : X \times X \to [0;1]$, which is reflexive, i.e. $\mu_R(x,x) = 1,\ \forall x \in X$.

If μ_R is the fuzzy preference relation in the set X, then for any pair of alternatives $x, y \in X$ the value $\mu_R(x,y)$ represents the measure of the implementation of the preference "*x is not worse than y*", or $x \geq y$. From the fact that $\mu_R(x,y) = 0$, follows from one of two statements: either $y = x$ or x and y are not comparable with each other to a positive degree. Reflexivity of this relation reflects the fact that any alternative is not worse than itself.

The fuzzy relation of preference given in the set X uniquely sets out three corresponding fuzzy relations:

- *indifference* $R^I\left(\mu_R^I\right)$;
- *quasi-equivalence* $R^e\left(\mu_R^e\right)$;
- *strict preference* $R^S\left(\mu_R^S\right)$.

These relations will be used to identify and analyze the properties of non-dominating alternatives in decision-making tasks.

By analogy with the usual relations, they can be defined as follows:

$$R^I = \left(X \times X \setminus R \bigcup R^{-1}\right) \bigcup \left(R \bigcap R^{-1}\right),$$

$$R^S = R \setminus R^{-1},$$

$$R^e = R \bigcap R^{-1},$$

where R^{-1} is the inverse to R relation, which is described by a membership function:

$$\mu_{R^{-1}}(x,y) = \mu_R(x,y), \qquad \forall (x,y) \in X \times X.$$

Using the definition of union, intersection and fuzzy sets difference, we obtain the following expressions to describe the membership functions of these relations.

1. Fuzzy relation of indifference

$$\mu_R^I(x,y) = \max\{1 - \max\{\mu_R(x,y), \mu_{\mathrm{R}}(y,x)\}, \min\{\mu_R(x,y), \mu_{\mathrm{R}}(y,x)\}\} =$$

$$= \max\left\{\min\left\{1-\mu_R\left(x,y\right),1-\mu_R\left(y,x\right)\right\},\min\left\{\mu_R\left(x,y\right),\mu_R\left(y,x\right)\right\}\right\}.$$

2. Fuzzy relation of quasi-equivalence

$$\mu_R^e(x,y) = \min\{\mu_R(x,y), \mu_{\mathrm{R}}(y,x)\}.$$

3. Fuzzy relation of strict preference

$$\mu_R^S(x,y) = \begin{cases} \mu_R(x,y) - \mu_R(y,x), \text{ if } \mu_R(x,y) \geq \mu_R(y,x), \\ 0, \text{ if } \mu_R(x,y) < \mu_R(y,x). \end{cases}$$

The meaning of these relationships can be explained by the example given below.

Example 5.4 (classical relation of preference). There are n functions $f_i : X \to R^1$, $i = 1, \dots, n$ given in the set X. Let's assign to the set X the relation of the preference R as follows: $x\,R\,y \Leftrightarrow f_i(x) \geq f_i(y), \ \forall \ i = 1,\dots,n$.

It is easy to notice that membership function of the relation R looks like as following:

$$\mu_R(x,y) = \begin{cases} 1, \text{if } f_i\left(x\right) \geq f_i\left(y\right), \forall i = \overline{1,n}, \\ 0 \ \text{ in other case.} \end{cases}$$

Note that in this relation, the preferences in the set X can be alternatives that cannot be compared, i.e. $R \cup R^{-1} \neq X \times X$, and there are alternatives x, y, for which the condition $(x,y) \notin R \cup R^{-1}$ is fulfilled. For example, alternatives x, y, for which $f_i(x) \geq f_i(y), \ \forall \ i \neq i_\circ$, but $f_{i_\circ}(x) < f_{i_\circ}(y)$.

Given the above definition,

$$\mu_R^e(x,y) = \begin{cases} 1, \text{if } f_i\left(x\right) = f_i\left(y\right), \forall i = \overline{1,n}, \\ 0 \ \text{ in other case.} \end{cases}$$

It should be emphasized that alternatives, which are not dominated in relation to this relation of preference, are called effective or optimal, according to Pareto, for functions $f_i(x)$, $i = 1, 2, \dots, n$.

Now let's consider some properties of fuzzy relations μ_R^e and μ_R^S.

I. Fuzzy relations μ_R^e and μ_R^I are reflexive and symmetric.

Indeed, $\mu_R^e(x,x)=\mu_R^I(x,x)=\mu_R(x,x)=1$ since the original relation μ_R is reflexive. The symmetry of these relations follows from their definitions.

II. The relation μ_R^S is anti-reflexive and anti-symmetric.

Really, $\mu_R^S(x,x)=0$ since the output IRP is reflexive, that is $\mu_R(x,x)=1$, $\forall x \in X$.

Let $\mu_R^S(x,y)>0$, that is $\mu_R(x,y)-\mu_R(y,x)>0$, then $\mu_R^S(y,x)=0$, and this is an evidence of the antisymmetry of this relation.

Now let`s show that when the output IRP μ_R in the set X belongs to the transitive one, then the fuzzy relations μ_R^e and μ_R^S are also transitive.

Theorem 5.2. If the IRP μ_R in the set X is transitive, then the corresponding fuzzy relation μ_R^e will also be transitive.

It is necessary to note that from this theorem and from the above investigated properties of the relation μ_R^e, it follows that under the conditions of the theorem it is a fuzzy relation of equivalence (reflexive, symmetric, transitive).

Proof

Assume that under the conditions of the theorem, the relation μ_R^e is not transitive. In other words, alternatives $x, y, z \in X$ can be found for which there will be fair the following inequality:

$$\mu_R^e(x,y)<\min\left\{\mu_R^e(x,z),\mu_R^e(z,y)\right\} . \tag{5.18}$$

Assume now that $\mu_R(x,y)\geq\mu_R(y,x)$, then from the definition of the relation μ_R^e we draw the conclusion: $\mu_R^e(x,y)=\mu_R(y,x)$. Using this equality, we write the inequality (5.18) in this form:

$$\mu_R(y,x)<\min\{\mu_R^e(y,z),\mu_R^e(z,x)\}. \tag{5.19}$$

Since the relation μ_R^e is symmetric, then inequality (5.19)

$$\mu_R^e(y,z)\leq\mu_R(y,z),$$

$$\mu_R^e(z,x)\leq\mu_R(z,x),$$

that is $\min\{\mu_R^e(y,z),\mu_R^e(z,x)\}\leq\min\{\mu_R(y,z),\mu_R(z,x)\}$, And $\min\{\mu_R^e(y,z),\mu_R^e(z,x)\}\le\min\{\mu_R(y,z),\mu_R(z,x)\}$ which contradicts the condition of the transitivity of the original relation at which $\mu_R(x,y)\ge\max_{z\in X}\min\{\mu_R(x,z),\mu_R(z,y)\}$

The case, when $\mu_R(y,x)\ge\mu_R(x,y)$, is proved in the same way.

It is possible to prove a similar statement for the relation of the strict preference μ_R^S.

Theorem 5.3. If the fuzzy relation of the preference μ_R in the set X is transitive, then the corresponding fuzzy relation of the strict preference μ_R^S will also be transitive.

Proof

Assume that under the conditions of this theorem, the relation μ_R^S is not transitive. This means that there are alternatives $x,y,z\in X$ for which such inequality is true:

$$\mu_R^S(x,y)<\min\{\mu_R^S(x,z),\mu_R^S(z,y)\}. \tag{5.20}$$

Since $\mu_R^S(x,y)>0,\quad\forall x,y\in X$, then

$$\mu_R^S(x,z)=\mu_R(x,z)-\mu_R(z,x)>0, \tag{5.21}$$

$$\mu_R^S(z,y)=\mu_R(z,y)-\mu_R(y,z)>0. \tag{5.22}$$

Consider two cases.

a) Let $\mu_R(x,y)\le\mu_R(y,x)$, then, given the transitivity of the relation μ_R, we can write the following inequality as:

$$\mu_R(y,x)\ge\mu_R(x,y)\ge\min\{\mu_R(x,z),\mu_R(z,y)\}. \tag{5.23}$$

In addition, from the transitivity of the relation μ_R and inequality (5.21) it follows that

$$\mu_R(x,z)\ge\mu_R(z,x)\ge\min\{\mu_R(z,y),\mu_R(y,x)\}. \tag{5.24}$$

Taking into account the relations (5.23) and (5.24), we come to the following conclusion:

$$\mu_R(x,z) \ge \min\{\mu_R(z,y), \mu_R(x,z)\}, \tag{5.25}$$

That is

$$\mu_R(x,z) > \mu_R(z,y), \tag{5.26}$$

and from (5.23) it follows that

$$\mu_R(x,y) \ge \mu_R(z,y). \tag{5.27}$$

Further, since the relation μ_R is transitive, then

$$\mu_R(y,z) \ge \min\{\mu_R(y,x), \mu_R(z,y)\} \ge \min\{\mu_R(x,y), \mu_R(z,y)\}.$$

Taking into account this inequality and the relation (5.27), we conclude that $\mu_R(y,z) \ge \mu_R(z,y)$, and this contradicts the statement (5.22).

Thus, we see that it follows from the condition (5.20) that it is impossible to perform such inequality:

$$\mu_R(x,y) \le \mu_R(y,x).$$

b) Suppose now that $\mu_R(x,y) > \mu_R(y,x)$, then

$$\mu_R^S(x,y) = \mu_R(x,y) - \mu_R(y,x) > 0,$$

and the inequality (5.20) can be written as follows:

$$\mu_R(x,y) - \mu_R(y,x) < \min\{[\mu_R(x,z) - \mu_R(z,x)], [\mu_R(z,y) - \mu_R(y,z)]\}. \tag{5.28}$$

Next, assume that $\mu_R(y,z) \ge \mu_R(y,x)$, then the function $\mu_R(y,z)$ in expression (5.28) can be replaced by $\mu_R(y,x)$, that is

$$\mu_R(x,y) - \mu_R(y,x) < \min\{[\mu_R(x,z) - \mu_R(z,x)], [\mu_R(z,y) - \mu_R(y,x)]\}. \tag{5.29}$$

When added to both parts of this inequality $\mu_R(y,x)$, this expression becomes of the following form:

$$\mu_R(x,y) < \min\{[\mu_R(x,z) + (\mu_R(y,x) - \mu_R(z,x))], \mu_R(z,y)\}. \tag{5.30}$$

Here two opportunities should be considered.

1) If $\mu_R(y,x)-\mu_R(z,x)\le 0$, then from (5.30) we obtain the following inequality:

$$\mu_R(x,y)<\min\{\mu_R(x,z),\mu_R(z,y)\},$$

and it contradicts the transitivity of the relation μ_R.

2) When $\mu_R(y,x)-\mu_R(z,x)>0$, then, taking into account transitivity, we can conclude that

$$\mu_R(x,y)>\mu_R(z,x)\ge\min\{\mu_R(z,y),\mu_R(y,x)\}.$$

This leads to the following inequality:

$$\mu_R(y,x)>\mu_R(z,y),$$

and it contradicts the fact that $\mu_R(y,z)\ge\mu_R(y,x)$.

So, we have demonstrated that when the condition is fulfilled: $\mu_R(x,y)>\mu_R(y,x)$, then from the expressions (5.21) and (5.22) there the inequality follows:

$$\mu_R(y,z)<\mu_R(y,x). \tag{5.31}$$

Similarly, it can be shown that when $\mu_R(x,y)>\mu_R(y,x)$, from the expressions (5.21) and (5.22), the given inequality follows:

$$\mu_R(z,x)<\mu_R(y,x). \tag{5.32}$$

Further, taking into account that

$$\mu_R(y,z)\ge\min\{\mu_R(y,x),\mu_R(x,z)\},$$

$$\mu_R(z,x)\ge\min\{\mu_R(z,y),\mu_R(y,x)\},$$

From the expressions (5.31) and (5.32) we obtain the following inequalities:

$$\mu_R(y,z)\ge\mu_R(x,z),$$

$$\mu_R(z,x)\ge\mu_R(z,y),$$

and they contradict the assumption (5.21) and (5.22).

The theorem has been proved.

5.4.2 Fuzzy subset of non-dominated alternatives

Now let's consider the rational choice of alternatives from the set *X*, in which the fuzzy relation of the preference *R*, its membership function is $\mu_R : X \times X \to [0;1]$.

As it was noted earlier, in those cases, where decision-making data about the situation is described in the form of a simple relation of preference, then the rationale can be considered as the choice of maximal (*non-dominated*) alternatives. Mathematically, this problem is reduced to the definition of a subset of *non-dominated* alternatives in a given set *X*.

Next, we will try to apply this approach to decision-making tasks when the relation of preference to the set of alternatives is not clearly described.

Consequently, assume we have the usual (clearly described) set of alternatives *X* and an fuzzy relation of non-strict preference μ_R, given in it, and fuzzy relation of the strict preference μ_R^S corresponding to it. Let's define a subset of *non-dominated* plural alternatives (X, μ_R). Note that since the initial relation of preference is fuzzy, then it is natural to expect that the corresponding subset of *non-dominated* alternatives will be fuzzy.

Given the definition of the relation of strict preference, for any alternatives $x, y \in X$, the value of $\mu_R^S(y,x)$ is the measure by which the alternative *x* will be dominated by the alternative *y*. Consequently, with respect to a fixed alternative $y \in X$, a function $\mu_R^S(y,x)$ defined in a set *X* can be considered a function of the membership of the fuzzy set of all alternatives *x* strictly dominated by the alternative *y*.

For example, the measure of the membership of an alternative x_0 to this set (corresponding to some fixed alternative *y*) is 0.3. This means that the x_0 is being dominated by the alternative *y* with a grade of 0.3. It is easy to understand that the set of "all" alternatives *x* not dominated by the alternative *y*, and this set is an addition to the set $\mu_R^S(y,x)$ in the set *X*, where its membership function can be written as follows:

$$1 - \mu_R^S(y,x),\ x \in X\ . \tag{5.33}$$

If, for example, $\mu_R^S(y,x) = 0{,}3$, then the alternative *x* is not dominated by the alternative y with a degree of 0,7. Obviously, to determine subsets of "all" alternatives in a set *X*, where each of them does not dominate any alternative of this set, it is necessary to take the intersection of the fuzzy sets described by the expression (5.33), with all alternatives $y \in X$.

Definition 5.4. Let X is the set of alternatives, μ_R is the relation of preference given on it. A fuzzy subset of *non-dominated alternatives* is referred to the intersection of fuzzy sets of the form corresponding to the expression (5.33) for all alternatives $y \in X$, that is,

$$\mu_R^{ND}(x) = \inf_{y \in X}\left[1 - \mu_R^S(y,x)\right], \quad x \in X, \tag{5.34}$$

or

$$\mu_R^{ND}(x) = 1 - \sup_{y \in X} \mu_R^S(y,x), \quad x \in X. \tag{5.35}$$

The function $\mu_R^{ND}(x)$ value represents the measure by which the alternative x will not dominate any alternative of the X set.

Let $\mu_R^{ND}(x_0) = \alpha$ for some alternative x_0. In this case, x_0 may be dominated by other alternatives, but with a degree not higher, than $(1-\alpha)$.

Indeed, at the same time

$$\sup_{y \in X} \mu_R^S(y, x_0) = 1 - \alpha,$$

and then

$$\mu_R^S(y, x_0) \le 1 - \alpha, \quad \forall y \in X.$$

Let`s now define an fuzzy subset of alternatives through the function of the membership of the output of the fuzzy preference μ_R. For this we will show that

$$\sup_{y \in X} \mu_R^S(y,x) = \sup_{y \in X}\left[\mu_R(y,x) - \mu_R(x,y)\right], \ \forall\, x \in X. \tag{5.36}$$

Indeed, let an alternative $x \in X$ be randomly chosen. Let`s introduce such sets

$$Y^1(x) = \{y | y \in X, \mu_{\mathrm{R}}(y,x) > \mu_R(x,y)\}, \tag{5.37}$$

$$Y^2(x) = \{y | y \in X, \mu_{\mathrm{R}}(y,x) \le \mu_R(x,y)\}. \tag{5.38}$$

Using the fact that $Y^1(x) \cup Y^2(x) = X$, for each alternative $x \in X$ we write equality (5.36) in the following form:

$$\sup_{y \in X} \mu_R^S(y,x) = \max\left\{\sup_{y \in Y^1(x)} \mu_R^S(y,x), \ \sup_{y \in Y^2(x)} \mu_R^S(y,x)\right\}. \tag{5.39}$$

Further, on the basis of the definition μ_R^S, we perform the transformation of the expression (5.39):

$$\sup_{y \in X} \mu_R^S(y,x) = \max\left\{ \sup_{y \in Y(x)} [\mu_R(y,x) - \mu_R(x,y)], 0 \right\} =$$

$$= \max\left\{ \sup_{y \in Y^1(x)} [\mu_R(y,y) - \mu_R(x,y)], \sup_{y \in Y^2(x)} [\mu_R(y,x) - \mu_R(x,y)] \right\} =$$

$$= \sup_{y \in X} [\mu_R(y,x) - \mu_R(x,y)].$$

Taking into account equality (5.36), we can describe a set of *non-dominated alternatives* using the following membership function:

$$\mu_R^{ND}(x) = 1 - \sup_{y \in X} \left[\mu_R(y,x) - \mu_R(x,y) \right], \quad x \in X. \tag{5.40}$$

Formula (5.40) may be useful in processing information presented in the form of a fuzzy preference relation to determine in a plurality of X subsets of non-dominated alternatives.

Since the value $\mu_R^{ND}(x)$ acts as a measure of the "non-dominance" of the alternative x, then it is rational, taking into account the fuzzy information given, to consider the choice of alternatives with the highest degree of belonging to the fuzzy set $\mu_R^{ND}(x)$, that is, those alternatives that give the value of a function $\mu_R^{ND}(x)$ closest to the following magnitude:

$$\sup_{x \in X} \mu_R^{ND}(x) = 1 - \inf_{y \in X} \sup_{y \in X} \left[\mu_R(y,x) - \mu_R(x,y) \right].$$

Alternatives for which the function $\mu_R^{ND}(x)$ reaches its upper edge, that is, the elements are of such a set:

$$X_{ND} = \left\{ x \middle| x \in X, \mu_R^{ND}(x) = \sup_{z \in X} \mu_R^{ND}(z) \right\},$$

We will call the maximal non-dominated alternatives of the set (X, μ_R).

Example 5.5. It is given a finite set: $X = \{x_1, x_2, x_3, x_4,\}$, and the fuzzy relation of the preference in it, function of membership of which has the following form:

$$\mu_R(x_i, x_j) = \begin{array}{c|c|c|c|c} & x_1 & x_2 & x_3 & x_4 \\ \hline x_1 & 1 & 0,2 & 0,3 & 0,8 \\ \hline x_2 & 0,5 & 1 & 0,2 & 0,1 \\ \hline x_3 & 0,6 & 0,4 & 1 & 0,5 \\ \hline x_4 & 0,1 & 0,1 & 0,3 & 1 \end{array}.$$

Let`s find a set of non-dominated alternatives of the set (X, μ_R).

According to the definition above,

$$\mu_R^S(x_i, x_j) = \begin{array}{c|c|c|c|c} & x_1 & x_2 & x_3 & x_4 \\ \hline x_1 & 0 & 0 & 0 & 0,7 \\ \hline x_2 & 0,3 & 0 & 0 & 0 \\ \hline x_3 & 0,3 & 0,2 & 0 & 0,2 \\ \hline x_4 & 0,1 & 0 & 0 & 0 \end{array},$$

Then

$$\mu_R^{ND}(x) = \begin{array}{c|c|c|c} x_1 & x_2 & x_3 & x_4 \\ \hline 0,7 & 0,8 & 1 & 0,3 \end{array}.$$

From this we see that the greatest degree of non-domination has an alternative x_3, and therefore its choice should be considered rational.

Definition 5.5. Let's call the *R*-relation in the set *X* a *linear* if the two alternatives to the set *X* are related to it or to its reverse relation.

That is, when the relation is linear, then there are no comparable alternatives in the set *X*. For ordinary relations, linearity means that the following statement is true:

$$R \cup R^{-1} = X \times X,$$

where R^{-1} – a converse to *R* relation, or using the terms of characterising functions

$$\mu_R(x, y) = 0 \quad \Rightarrow \quad \mu_R(y, x) = 1.$$

If there is a fuzzy relation, we can clearly identify only the complete lack of linearity, that is, the fuzzy relation μ_R will not be linear if and only if there are alternatives $x, y \in X$ for which the following equality is fulfilled:

$$\mu_R(x, y) = \mu_R(y, x) = 0,$$

where $\mu_R(x, y)$ – the function of the membership of this fuzzy relation.

Consequently, the property of linearity with respect to the fuzzy relation can be understood more widely.

Definition 5.6. Let λ be some number on the interval $[0;1]$. A fuzzy relation μ_R will be referred to as *λ-linear* if its membership function has the following property:

$$\max\{\mu_R(x,y),\mu_R(y,x)\}>\lambda,\quad \forall\ x,y\in X. \tag{5.41}$$

Thus, if, for example, the fuzzy order is a 0.7-linear relation, then from each of two alternatives one won't be worse than another with a degree more than 0.7.

Definition 5.7. A well-known relation is called a *strongly linear relation*, if its membership function satisfies the following condition:

$$\max\{\mu_R(x,y),\ \mu_R(y,x)\}=1,\quad \forall\ x,y\in X \tag{5.42}$$

In another way, this property can be defined as follows:

$$\mu_R(x,y)\geq\mu_R(y,x)\ \Rightarrow\ \mu_R(x,y)=1. \tag{5.43}$$

Let`s show that strong linearity corresponds to the following condition:

$$\mu_R(x,y)=1-\mu_R^S(y,x),\quad \forall\ x,y\in X, \tag{5.44}$$

where μ_R^S is the corresponding fuzzy relation of the strict preference.

Really, when condition (5.42) is fulfilled, taking into account the definition of the relation of the strict preference μ_R^S, we conclude that $\mu_R^S(y,x)=0$ and $\mu_R^S(x,y)=1$, that is, the condition (5.43) is also fulfilled. On the other hand, if the condition satisfied (5.43) and, moreover, the inequality $\mu_R(x,y)\geq\mu_R(y,x)$ is true, then $\mu_R^S(y,x)=0$ and $\mu_R(x,y)=1$ that is, the condition (5.42) also holds.

Let`s explain the meaning of strong linearity. For example, alternative x is better than alternative y with degree 1 ($x\succ y$). Then $(x,y)\notin R^{-1},(x,y)\notin R^{-1}$ y cannot be better than x with no positive degree. If there is relation $y\succ x$, then $(x,y)\in R^{-1}$, that is $y\succ x$ with degree 1. If x is better than y with degree α ($x\underset{\alpha}{\succ}y$), then preference with the degree $(1-\alpha)$ prevails: $y\succ x$. Thus, in its content strong linearity is similar to the linearity of the ordinary relation to great extent.

Strongly linear relations have the following properties:

1. $\mu_R(x_1,x_2)=1,\quad \mu_R(x_2,x_1)=0.$

2. The relations R^e and R^I, corresponding to a strongly linear relations, coincide.

Indeed, when for some pairs of alternatives $(x, y) \in X$ the condition is satisfied: $\mu_R(x, y) \geq \mu_R(y, x)$, then, from the definition of strong linearity follows that equality: $\mu_R(x, y) = 1$, and according to the definition of the relation μ_R^I, we conclude that $\mu_R^I(x, y) = \mu_R(x, y)$. As a result of the symmetry of the relation being μ_R^I when $\mu_R(x, y) \leq \mu_R(y, x)$, then

$$\mu_R^I(x, y) = \min\{\mu_R(x, y), \mu_R(y, x)\} = \mu_R^e(x, y).$$

Definition 5.8. The fuzzy relation of preference is called *weakly linear relation*, if it has the following property:

$$\mu_R(x, y) = 0 \Rightarrow \mu_R(y, x) > 0, \ \forall\, x, y \in X.$$

Here are examples of linear fuzzy relations.

Let X be a set consisting of 4 elements, then an fuzzy relation with such membership function:

$$\mu_R(x_i, x_j) = \begin{pmatrix} 1 & 0{,}55 & 0{,}6 & 0 \\ 0 & 1 & 0{,}3 & 1 \\ 0{,}2 & 0{,}6 & 1 & 0{,}4 \\ 0{,}8 & 1 & 0{,}7 & 1 \end{pmatrix},$$

will be 0,5 linear.

The relation described by the membership function of this form:

$$\mu_R(x_i, y_j) = \begin{pmatrix} 1 & 0{,}2 & 1 & 0 \\ 1 & 1 & 1 & 0{,}5 \\ 0 & 0{,}8 & 1 & 1 \\ 1 & 1 & 0{,}3 & 1 \end{pmatrix},$$

will be strongly linear.

5.4.3 Clearly non-dominated alternatives and their properties

In this subsection, we consider the problems in which the set of non-dominated alternatives is a normal fuzzy subset of the universal set X, that is, the membership function of this subset has the following property:

$$\sup_{x \in X} \mu_R^{ND}(x) = 1. \quad (5.45)$$

In this case, for alternative from the set of maximal non-dominated alternatives X^{ND}, the following condition is fulfilled: $\mu_R^{ND}(x) = 1$ that is, the measure of the non-dominance of each of them is equal to 1.

In other words, for each alternative $x \in X^{ND}$ and any alternative $y \in X$, the following equality is fulfilled: $\mu_R^S(y, x) = 0$, that is, no alternative dominates with a positive degree the given alternative *x*.

Therefore, we will call these alternatives *clearly* non-dominated. Denote the set of such alternatives as X^{CND}. Thus,

$$X^{\text{CND}} = \left\{ x \in X \middle| \mu_R^{ND}(x) = 1 \right\}. \quad (5.46)$$

As follows from the definition of the sets X^{CND} and μ_R^{ND}, for each clearly non-dominated alternative, the following condition is fulfilled:

$$\sup_{y \in X} \mu_R^S(y, x) = 0, \quad \forall x \in X^{\text{ЧНД}}, \quad (5.47)$$

where μ_R^S is the fuzzy relation of the strict preference, which corresponds to the relation μ_R.

This leads to the conclusion that for any alternatives $x_1, x_2 \in X^{CND}$ the following equality will be valid:

$$\mu_R^S(x_1, x_2) = \mu_R^S(x_2, x_1) = 0. \quad (5.48)$$

From the definition it follows that equality (5.48) is equivalent to equality:

$$\mu_R(x_1, x_2) = \mu_R(x_2, x_1),$$

but then

$$\mu_R^I(x_1, x_2) = \max\left\{1 - \mu_R(x_1, x_2), \mu_R(x_1, x_2)\right\} \geq 0{,}5.$$

In other words, any two clearly non-dominated alternatives are related to the relation of indifference with a degree more than 0.5.

And the corresponding fuzzy equivalence μ_R^e relation will be defined as follows:

$$\mu_R^e(x_1, x_2) = \mu_R(x_1, x_2) = \mu_R(x_2, x_1), \ \forall x_1, x_2 \in X^{CND}. \quad (5.49)$$

When there are random fuzzy relations of preference, it may turn out that $\mu_R^e(x_1, x_2) = 0$ for some alternatives $x_1, x_2 \in X^{CND}$, that is, with no positive additive measure, these alternatives will not be equivalent. Note, that then $\mu_R(x_1, x_2) = 0$, that is, x_1 and x_2 are not comparable with each other. However, this does not apply to linear relations.

5.5 Making decisions on the existence of several relations of preference in a set of alternatives

Consider the problem in which the set of alternatives *X* is given, and each alternative of this set is characterized by several features, the numbers of which is $j = 1,..,m$. The information on pairwise comparison of alternatives is given in the form of preference relations R_j, $j = 1, 2, \ldots m$. Thus, we have *m* preference relations in the set *X*. Our goal is to make a rational choice of alternatives from the set $(X, R_1, R_2 \ldots R_m)$ on the basis of this information.

Let`s deal first with a situation where relations are described by numerical utility functions $f_j : X \to R^1$, $j = 1, \ldots, m$, where R^1 is the numerical axis. The value of a function $f_j(x)$ can be considered a numerical evaluation of the alternative *x* on the basis of feature *j*, $j = 1, 2, \ldots m$. The preference by the feature *j* is given to the alternative, which is characterized by a higher value $f_j(x)$. The task is to choose an alternative that has the highest ratings for all features. It is rational in this case to consider the choice of the alternative x_0, which has the following property:

$$\text{if } f_j(y) \ge f_j(x_0),\ j = 1, 2, \ldots, m, \text{ then } f_j(y) = f_j(x_0),\ j = 1, 2, \ldots, m. \quad (5.50)$$

Such alternatives in multi-criteria optimization are called effective.

It is easy to see that each function $f_j(x)$, $j = 1, 2, \ldots m$, describes the usual preference relation in the set of alternatives in the following way:

$$R_j = \{(x, y) \mid x, y \in X,\ f_j(x) \ge f_j(y)\}. \quad (5.51)$$

Let`s $Q_1 = \bigcap_{j=1}^{m} R_j$. Let`s make sure that the set of all (non-dominated) alternatives in the set (X, Q_1) coincides with the set of effective alternatives for a set of functions f_j, $j = 1, 2, \ldots m$.

Assume that x_0 is an alternative that does not dominate in the set (X, Q_1). This means that for any alternative $y \in X$, the following condition is true:

$$(y, x_0) \notin Q_1^S, \tag{5.52}$$

where Q_1^S – the relation of the strict preference, corresponding to the relation Q_1, it has the following form:

$$Q_1^S = \{(x, y) \mid x, y \in X, f_j(x) \geq f_j(y),\ j = 1, \ldots, m,\ \exists j_0\colon\ f_{j_0}(x) > f_{j_0}(y)\}. \tag{5.53}$$

Hence, taking into account the condition (5.52), we conclude that there is a property (5.50), that is x_0 – an effective alternative to the function $f_j(x)$, $j = 1, 2, \ldots m$.

We can also show the opposite, that is, any effective alternative for a set of functions $f_j(x)$, $j = 1, 2, \ldots m$ is not dominated in the set (X, Q_1). Thus, in order to find the set of effective alternatives, one can take instead of a set of relations R_j, $j = 1, 2, \ldots m$ their intersection Q_1 and find a set of non-dominated alternatives in the set (X, Q_1). Let's now write the intersection of relations R_j in a different form.

Let

$$\mu_j(x, y) = \begin{cases} 1, & if\ (x, y) \in R_j, \\ 0, & if\ (x, y) \notin R_j, \end{cases} \tag{5.54}$$

where $\mu_j(x, y)$ – is the membership function of the relation R_j, $j = 1, 2, \ldots m$ then the intersection of these relations corresponds to the following membership function:

$$\mu_{Q_1}(x, y) = \min\{\mu_1(x, y), \ldots, \mu_m(x, y)\}. \tag{5.55}$$

It acts as an analogue of the convolution of criteria f_j : $F(x) = \min\limits_{j=1,..,m} \lambda_j f_j$, in multicriteria decision-making problems. Here numbers λ_i are the coefficients of the relative importance of the criteria. In the convolution (5.55) $\lambda_j = 1$, $j = 1, \ldots, m$, which corresponds to the situation, where all the above relations are equally important to consider when choosing alternatives. When such relations differ in importance of the corresponding features, on the basis of which the alternatives are compared, then various by magnitude coefficients λ_j can be used in the convolution (5.55). At the same time, we must consider the initial relations as fuzzy, that is, in determining the membership function (5.54) the numbers 0 and 1 must be considered as extreme points of the single interval of possible values of the membership degree.

As a result of the convolution of the initial relations R_j with the coefficients λ_j corresponding to the condition: $\sum_{j=1}^{m}\lambda_j = 1$, we obtain the membership function, which looks like this:

$$\mu_{Q_1}(x,y) = \min\{\lambda_1\mu_1(x,y),\ldots,\lambda_m\mu_m(x,y)\}, \tag{5.56}$$

that is, the function of membership of the fuzzy preference relation. But this relation will not be reflexive. Therefore, it does not belong to the preferences in the sense of the definition of paragraph 5.4.1, and therefore the convolution is described as inconvenient for use when it is necessary to take into account the importance of the relations presented.

That is why we consider the convolution of output relation of another form:

$$\mu_{Q_2}(x,y) = \sum_{j=1}^{m}\lambda_j\mu_j(x,y). \tag{5.57}$$

Note that the result obtained after convolution (5.38) of the usual relations R_j, the fuzzy relation μ_{Q_2} will be reflexive since such are the initial relations.

Let all outgoing relations of preference be the same in importance. In the curl (5.57) of this case, they correspond to following values of the weighted coefficients: $\lambda_j = \frac{1}{m}$, $j = 1, 2, \ldots m$. Let`s find a subset of alternatives, non-dominated by a set (X, Q_2), using the definition in 5.4.2, as follows:

$$\mu_{Q_2}^{ND}(x) = 1 - \frac{1}{m}\sup_{y\in X}\left[\mu_j(y,x) - \mu_j(x,y)\right],\ \forall\, x \in X. \tag{5.58}$$

Denote a subset of clearly non-dominated alternatives in a set (X, μ_{θ_1}) as X_1^{CND} and X_2^{CND} as the corresponding subset in (X, μ_{θ_2}). Let`s set that $X_2^{\text{CND}} \subset X_1^{\text{CND}}$. Indeed, let $x_0 \in X_2^{\text{CND}}$, then, in accordance with the definition of a clearly non-dominated alternative, and taking into account formula (5.58), we can conclude that

$$\sup_{y\in X}\sum_{j=1}^{m}\left[\mu_j(y,x_0) - \mu_j(x_{0,y})\right] = 0$$

or

$$\sum_{j=1}^{m}\left[\mu_j(y,x_0)-\mu_j(x_0,y)\right]\le 0, \qquad (5.59)$$

for all alternatives $y \in X$.

Assume that $x_0 \notin X_1^{\mathrm{CND}}$. Then, in accordance with the property (5.50) and definition (5.54), we see that there is such an alternative $y \in X$, for which $\mu_j(y,x_0)=1,\ \ j=1,2,\dots m$, with respect to some index j_0, equality is performed: $\mu_{j_0}(x_0,y)=0$. But then, in relation to the alternative y, there will be no fair inequality (5.59). It follows that $x_0 \in X_1^{\mathrm{CND}}$, and, accordingly, $X_2^{\mathrm{CND}} \subset X_1^{\mathrm{CND}}$.

Remark: The set X_2^{CND} does not cover all the effective alter natives for function $f_j,\ \ j=1,2,\dots m$, i.e. doesn't coincide with the set $X_1^{\mathrm{CND.}}$ Though, we can show that each effective alternative, i.e. each element $x \in X_1^{\mathrm{CND}}$ have an additive degree of belonging to the set $\mu_{Q_2}^{ND}$, i.e.

$$X_1^{CND} \subseteq \mathrm{supp}\mu_{Q_2}^{ND}.$$

Indeed, when equality is satisfied with respect to any alternative $x \in X$ $\mu_{Q_2}^{ND}(x)=0$, then, based on the definition (5.58), we find that in the set X we can select an alternative y, for which

$$\mu_j(y,x)-\mu_j(x,y)=1, \quad j=1,2,\dots m,$$

That is $\mu_j(y,x)=1$ and $\mu_j(x,y)=0$, $j=1,2,\dots m$. This means that the alternative y is dominated by the alternative x, that is $f_j(y)>f_j(x)$, $j=1,2,\dots m$, and therefore the alternative x can not be effective for the function set f_j.

The function $\mu_{Q_2}^{ND}$ arranges alternatives by the degree of their non-domination.

For example, if $\mu_{Q_2}^{ND}(x)=3/4$ and there is an alternative $y \in X$ a better than the alternative of x by any of the two features, then, at least according to one of the other signs, it strictly prevails with the alternative y.

If we take the intersection of the sets X_1^{CND} and $\mu_{Q_2}^{ND}$ then we obtain the appropriate ordering in the set of effective alternatives, on the basis of which it is possible to make a choice among them.

Consequently, the application of the convolution (5.57) of the original normal relations to solving the decision-making problem in the set of functions allows us to obtain additional information on the relative degree of non-domination of effective alternatives, thereby narrowing the class of rational choices to such a set:

$$X^{CND} = \left\{ x \middle| x \in X, \mu_{Q_2}^{ND}(x) = \sup_{x' \in X_2^{ND}} \mu_{Q_2}^{ND}(x') \right\}.$$

In the general task, when the set of alternatives is given as the set of *m* fuzzy preference relations R_j, $j = 1, 2, \dots m$ and the coefficients of the relative weight of these relations λ_j, $j = 1, 2, \dots m$, are given, we can act in the same way as in the previous case.

Let`s now formulate the decision-making algorithm for several given relations of preference over the set of alternatives.

1. Construct a fuzzy relation Q_1 (intersection of initial relations):

$$\mu_{Q_1}(x, y) = \min\{\mu_1(x, y), \dots, \mu_m(x, y)\}.$$

Next define the fuzzy subset of non-dominated alternatives in the set (X, μ_{Q_1}) with the following formula:

$$\mu_{Q_1}^{ND}(x) = 1 - \sup_{y \in X} \left[\mu_{Q_1}(x, y) - \mu_{Q_2}(y, x) \right].$$

2. Create a fuzzy relation Q_2 [the convolution of the relations of type (5.57)]:

$$\mu_{Q_2}(x, y) = \sum_{j=1}^{m} \lambda_j \mu_j(x, y)$$

and define a fuzzy subset of non-dominated alternatives in a set (X, μ_{Q_2}):

$$\mu_{Q_2}^{ND}(x) = 1 - \sup_{y \in X} \left[\mu_{Q_2}(x, y) - \mu_{Q_2}(y, x) \right].$$

3. Find the intersection of sets $\mu_{Q_1}^{ND}$ and $\mu_{Q_2}^{ND}$ the following rule:

$$\mu^{ND}(x) = \min\left\{ \mu_{Q_1}^{ND}(x), \mu_{Q_2}^{ND}(x) \right\}.$$

4. Consider the rational choice of alternatives from the following set:

$$X^{ND} = \left\{ x \in X \middle| \mu^{ND}(x) = \sup_{x' \in X} \mu^{ND}(x') \right\}.$$

It should be noted here that, depending on the type of task, it is reasonable not only to consider alternatives from the set of X^{ND}, but in one sense or another, weakly (or not very strongly) dominated alternatives, that is, those which degree of belonging to the set $\mu^{н.д}$ of lower than a given one.

Example 5.6. Let the set $X = \{x_1, x_2, x_3, x_4\}$, and there three clear relations of preference with the same significance are given, namely:

$R_1 =$

	x_1	x_2	x_3	x_4
x_1	1	1	0	0
x_2	1	1	1	0
x_3	0	0	1	0
x_4	0	1	0	1

$R_2 =$

	x_1	x_2	x_3	x_4
x_1	1	1	0	0
x_2	0	1	1	0
x_3	0	0	1	0
x_4	0	0	1	1

$R_3 =$

	x_1	x_2	x_3	x_4
x_1	1	1	0	0
x_2	1	1	1	0
x_3	0	0	1	0
x_4	0	1	1	1

It is necessary to make a rational choice of the alternative from the set X on their basis.

Solving

Since the relations of preference have the same significance, we state that the coefficients of relative weight $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$.

1. Build a relation: $Q_1 = \lambda_1 R_1 \cap \lambda_2 R_2 \cap \lambda_3 R_3$, for our data, it becomes the following:

$\mu_{Q_1}(x_i, x_j) =$

	x_1	x_2	x_3	x_4
x_1	1/3	1/3	0	0
x_2	0	1/3	1/3	0
x_3	0	0	1/3	0
x_4	0	0	0	1/3

Find the relation of strict preference, that is

$\mu_{Q_1}^{s}(x_i, x_j) =$

	x_1	x_2	x_3	x_4
x_1	0	1/3	0	0
x_2	0	0	1/3	0
x_3	0	0	0	0
x_4	0	0	0	0

Next, find a subset of non-dominated alternatives in the set (X, μ_{Q_1}):

$\mu_{Q_1}^{ND}(x) =$

x_1	x_2	x_3	x_4
1	2/3	2/3	1

2. Build a relation:

$$Q_2 = \frac{1}{3}\left(\mu_1(x_i, x_j) + \mu_2(x_i, x_j) + \mu_3(x_i, x_j)\right)$$

which, according to our data, takes the form of:

$$\mu_{Q_2}(x_i, x_j) = \begin{array}{c|c|c|c|c} & x_1 & x_2 & x_3 & x_4 \\ \hline x_1 & 1 & 1 & 0 & 0 \\ \hline x_2 & 2/3 & 1 & 1 & 0 \\ \hline x_3 & 0 & 0 & 1 & 0 \\ \hline x_4 & 0 & 1/3 & 2/3 & 1 \end{array}$$

Record the corresponding relation of the strict preference, that is

$$\mu_{Q_2}^{S}(x_i, x_j) = \begin{array}{c|c|c|c|c} & x_1 & x_2 & x_3 & x_4 \\ \hline x_1 & 0 & 1/3 & 0 & 0 \\ \hline x_2 & 0 & 0 & 1 & 0 \\ \hline x_3 & 0 & 0 & 0 & 0 \\ \hline x_4 & 0 & 1/3 & 2/3 & 0 \end{array}$$

Find a subset of non-dominated alternatives in the set (X, μ_{Q_2}):

$$\mu_{Q_2}^{ND}(x) = \begin{array}{c|c|c|c} x_1 & x_2 & x_3 & x_4 \\ \hline 1 & 2/3 & 0 & 1 \end{array}.$$

3. The set of non-dominated alternatives is the intersection of sets $\mu_{Q_1}^{ND}$ and $\mu_{Q_2}^{ND}$, that is

$$\mu^{ND}(x) = \begin{array}{c|c|c|c} x_1 & x_2 & x_3 & x_4 \\ \hline 1 & 2/3 & 0 & 1 \end{array}.$$

Conclude, that in the above example, it is considered rational to choose alternatives x_1 and x_4, which have the maximum degree of non-dominance.

Example 5.7. Let the set $X = \{x_1, x_2, x_3\}$. There are two fuzzy relations between the preferences R_1 and R_2, the first of which is significant, half as less as the latter, in particular

$$R_1 = \begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_1 & 1 & 0{,}5 & 0{,}3 \\ x_2 & 0 & 1 & 0{,}8 \\ x_3 & 1 & 0{,}5 & 1 \end{array}, \qquad R_2 = \begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_1 & 1 & 0{,}1 & 0 \\ x_2 & 0{,}3 & 1 & 1 \\ x_3 & 1 & 0{,}5 & 1 \end{array}$$

It is necessary to make a rational choice of the alternative from the set of X based on the given relations of preference.

Solving

1. Construct the relation: $Q_1 = \lambda_1 R_1 \cap \lambda_2 R_2$ taking into account $\lambda_1 = 0{,}33$ and $\lambda_2 = 0{,}67$, it takes on the following form:

$$\mu_{Q_1}\left(x_i, x_j\right) = \begin{pmatrix} 0{,}33 & 0{,}067 & 0 \\ 0 & 0{,}33 & 0{,}264 \\ 0{,}33 & 0{,}165 & 0{,}33 \end{pmatrix},$$

and a corresponding strict preference relation

$$\mu_{Q_1}^{s}\left(x_i, x_j\right) = \begin{pmatrix} 0 & 0{,}033 & 0 \\ 0 & 0 & 0{,}167 \\ 0{,}33 & 0 & 0 \end{pmatrix}.$$

Find a subset of non-dominated alternatives in the set $\left(X, \mu_{Q_1}\right)$. Its membership function

$$\mu_{Q_1}^{ND}\left(x\right) = \begin{array}{c|c|c} x_1 & x_2 & x_3 \\ \hline 0{,}67 & 0{,}967 & 0{,}833 \end{array}.$$

2. Construct a relation: $Q_2 = \lambda_1 \mu_1\left(x_i, x_j\right) + \lambda_2 \mu_2\left(x_i, x_j\right)$. Its membership function:

$$\mu_{Q_2}\left(x_i, x_j\right) = \begin{pmatrix} 1 & 0{,}367 & 0{,}2 \\ 0{,}2 & 1 & 0{,}867 \\ 1 & 0{,}5 & 1 \end{pmatrix},$$

and the corresponding relation of the strict preference looks as follows:

$$\mu_{Q_2}^{S}\left(x_i, x_j\right) = \begin{pmatrix} 0 & 0{,}167 & 0 \\ 0 & 0 & 0{,}367 \\ 0{,}8 & 0 & 0 \end{pmatrix}.$$

Find a subset of non-dominated alternatives in the set (X, μ_{Q_2}):

$$\mu_{Q_2}^{ND}(x) = \frac{x_1}{0,2} \left| \frac{x_2}{0,833} \right| \frac{x_3}{0,633}.$$

The initial set of non-dominated alternatives has the following membership function:

$$\mu^{ND}(x) = \frac{x_1}{0,2} \left| \frac{x_2}{0,833} \right| \frac{x_3}{0,633}.$$

The maximum degree of non-dominance is characterized by the alternative x_2, so its choice can be considered rational.

5.6 Relation of preference in fuzzy set of alternatives

Let`s consider now the case when the subset of admissible alternatives is also fuzzy.

Let X be a universal set of alternatives, and there is an fuzzy subset of admissible alternatives, whose membership function is $\nu : X \to [0;1]$, as well as the fuzzy relation of the preference with the membership function $\mu_R(x, y)$, are given on it.

In case, when the set of permissible alternatives is a normal set, the choice of a rational alternative occurs only depending on the fuzzy relations given to it in the preferences. But now we need to take into account also the degree of affiliation of the alternative to the set of permissible alternatives, that is, the preference should be given to those of them that are more important to the function $\nu(x)$.

This requirement can be taken into account as follows:

Let`s define the relation of the preference generated by the function as follows:

$$\mu_A(x) = \begin{cases} 1, \text{ if } \nu(x) \geq \nu(y), \\ 1, \text{ if } \nu(x) < \nu(y). \end{cases}$$

Now the original task is reduced to the statement studied in the previous paragraph, and to solve it, you can use the procedure described there.

5.7 Making decisions when preference is given in a set of properties

Given a set of alternatives X and a set of properties (or experts) of P. Each $x \in X$ is to some extent inherent in every property of the set P. For each fixed sign $p \in P$ is known the fuzzy relation of the preference of φ on the set X, that is, we know the function of membership $\varphi : X \times X \times P \to [0;1]$. Its value $\varphi(x_1, x_2, p)$ represents the degree of preference of the alternative x_1 over the alternative x_2 on the sign of p. If P is the set of experts, then $\varphi(x_1, x_2, p)$ is the relation of the preference over the set of alternatives proposed by the expert p. Thus, the function φ describes a family of fuzzy preferences on the set X relative to the parameter p.

The elements of the set P differ in their importance, and the fuzzy relation of $\mu : P \times P \to [0;1]$ describes the importance of the signs, in particular, the value $\mu(p_1, p_2)$ shows the degree to which the sign p_1 is considered as no less important than the sign p_2.

The task is to rationally choose an alternative from the set X based on the information described above.

Consider one of the possible approaches to solving this problem.

Denote by $\varphi^{ND}(x, p)$ the fuzzy subset of the non-dominated alternatives, which corresponds to the fuzzy relation of the preference $\varphi(x_1, x_2, p)$ for the fixed property $p \in P$:

$$\varphi^{ND}(x, p) = 1 - \sup_{y \in X}\left[\varphi(y, x, p) - \varphi(x, y, p)\right]. \tag{5.60}$$

If the alternatives were compared on the basis p, then it would be rational to consider the choice of those that provide the greatest value of the membership function $\varphi^{ND}(x, p)$ (the degree of non-domination) on the set X. But in this case, it is necessary to choose an alternative taking into account a set of features that differ in their importance.

For a fixed alternative $x_0 \in X$, the function $\varphi^{ND}(x_0, p)$ describes a fuzzy subset of properties, by which it is not dominated. It is clear that when for two alternatives x_1 and x_2 the fuzzy set $\varphi^{ND}(x_1, p)$ is "no less important" than the fuzzy set of properties $\varphi^{ND}(x_2, p)$, then the alternative x_1 should be considered as less acceptable than the alternative x_2. Thus, the situation in this case is similar to that which was considered in the analysis of the problem of fuzzy mathematical programming.

Consequently, now it is necessary to generalize the given fuzzy subset of the set P and to assume the received fuzzy relation to the resultant preference relation on the set of alternatives X.

This relation, generated by function $\varphi^{ND}(x,p)$ and fuzzy relation μ, will be determined by the following formula:

$$\eta(x_1,x_2)=\sup_{(p_1,p_2)\in P^2}\min\left\{\varphi^{ND}(x_1,p_1),\varphi^{ND}(x_2,p_2),\mu^{ND}(p_1,p_2)\right\}. \qquad (5.61)$$

This fuzzy preference relation can be considered as the result of a "convolution" of the family of fuzzy relations of $\varphi(x_1,x_2,p)$ into a single test fuzzy preference relation which takes into account information about the relative importance of the criteria given in the form of a fuzzy relation of preference.

Thus, the construction of the fuzzy relation of the preference of η to the original problem of choice is reduced to the problem of choice with a single preference relation. To solve it, it is enough to determine the corrected fuzzy set of non-dominated alternatives corresponding to the relation η and select those that give the maximum of the function $\eta^{ND}(x)$.

Consider the tasks that illustrate the described approach.

Example 5.8 (a choice based on clear relations). Let the set of alternatives be given: $X=\{x_1,x_2,x_3,x_4\}$, which are compared with each other according to the three signs A, B, C. The results of the comparison are described by the following matrices of relations of non-strict preference:

on the basis of A

	x_1	x_2	x_3	x_4
x_1	1	0	1	1
x_2	1	1	1	1
x_3	0	0	1	0
x_4	1	0	1	1

on the basis of B

	x_1	x_2	x_3	x_4
x_1	1	1	0	1
x_2	0	1	1	0
x_3	0	0	1	0
x_4	0	1	1	1

on the basis of C

	x_1	x_2	x_3	x_4
x_1	1	0	1	1
x_2	0	1	0	1
x_3	1	0	1	1
x_4	0	0	0	1

The relation of the relative importance of properties is described by a matrix of the following form:

$$\begin{array}{c|ccc} & A & B & C \\ \hline A & 1 & 1 & 1 \\ B & 1 & 1 & 1 \\ C & 0 & 0 & 1. \end{array}$$

From this matrix it is clear that properties *A* and *B* are equivalent to each other and each of them is more important than the property *C*.

Guided by the described approach, we define a set of alternatives that do not dominate each of the attributes, resulting in the following membership functions:

$$\varphi^{ND}(x,A)=\frac{x_1 \mid x_2 \mid x_3 \mid x_4}{0 \mid 1 \mid 0 \mid 0};$$

$$\varphi^{ND}(x,B)=\frac{x_1 \mid x_2 \mid x_3 \mid x_4}{1 \mid 0 \mid 0 \mid 0};$$

$$\varphi^{ND}(x,C)=\frac{x_1 \mid x_2 \mid x_3 \mid x_4}{1 \mid 1 \mid 1 \mid 0}.$$

Consequently, the following are not dominant:

- on the basis of A, the alternative x_2;
- on the basis of B, the alternative x_1;
- on the basis of C, the alternative x_1, x_2, x_3.

Further, by the formula (5.61) we have the following matrix of the resultant preference relation on the set of alternatives *X:*

$$\eta(x_i,x_j)=\begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4 \\ x_1 & 1 & 1 & 1 & 0 \\ x_2 & 1 & 1 & 1 & 0 \\ x_3 & 0 & 0 & 1 & 0 \\ x_4 & 0 & 0 & 0 & 0 \end{array}.$$

And according to the formula (5.60) we have the corresponding set of undamaged alternatives (unmodified):

$$\eta^{ND}(x,A) = \begin{array}{c|c|c|c} x_1 & x_2 & x_3 & x_4 \\ \hline 1 & 1 & 0 & 1 \end{array}.$$

Finally, using the formula: $\eta^{ND}(x_i) = \min\left\{\eta^{ND}(x_i), \eta^{ND}(x_i, x_i)\right\}$, we find the corrected set of non-matched alternatives:

$$\eta^{ND}_{cor}(x_i) = \begin{array}{c|c|c|c} x_1 & x_2 & x_3 & x_4 \\ \hline 1 & 1 & 0 & 0 \end{array}.$$

Thus, we conclude that the choice of alternative x_1 or x_2 should be considered rational in this task. Note that these alternatives are not dominated by features *A* and *B,* which are most (equally) important.

Example 5.9 (choice on the basis of fuzzy relations). Consider one of the typical decision-making tasks.

Assume that a Company Director is considering four further company developing projects for enterprises *A*, *B*, *C*, *D*. S/he should choose to implement only one of them. Aimed at this, s/he invited four experts: E_1, E_2, E_3, E_4, whose positions s/he treats in different way. In particular, s/he concurs one expert's conclusions more attentively and respectfully than to the views of others. The relative preferences of the positions of the experts are described by the following matrix of fuzzy relation "not less important":

	E_1	E_2	E_3	E_4
E_1	1	0,4	0,6	0
E_2	1	1	0,8	1
E_3	0,2	1	1	1
E_4	0,8	0	0	1

According to the experts, the relation of preferences between the projects is described by the membership functions, which are as following:

E_1	*A*	*B*	*C*	*D*
A	1	0,8	1	0
B	0	1	0,2	1
C	0	0,8	1	0
D	0	0	0	1

E_2	*A*	*B*	*C*	*D*
A	1	0,4	0,5	0,3
B	0,8	1	0,8	0,8
C	0,5	1	1	0
D	0,8	0	0	1

E_3	A	B	C	D
A	1	0	0,8	0
B	0	1	0	0
C	0,1	0	1	0,4
D	1	0	1	1

E_4	A	B	C	D
A	1	1	0,9	0
B	0	1	1	1
C	0,4	0	1	0
D	0	0	0	1

Solving

Find fuzzy sets of non-dominated alternativesfor each of the property:

	A	B	C	D
$\varphi^{ND}(\cdot, E_1)$	1	0,2	0	0
$\varphi^{ND}(\cdot, E_2)$	0,3	1	0,5	0,2
$\varphi^{ND}(\cdot, E_3)$	0	0	0,3	1
$\varphi^{ND}(\cdot, E_4)$	1	0	0	0

Next, we seek for a fuzzy preference relation η, defined on a set of functions φ^{ND} and a fuzzy relation μ, in particular:

	A	B	C	D
A	1	0,4	0,4	1
B	1	1	0,5	0,8
C	0,5	0,5	0,5	0,5
D	1	1	0,5	1

Finally, we determine the fuzzy set of non-dominated alternatives corresponding to the relation η ,which membership function has the following form:

$$\eta^{ND} = \frac{A \;|\; B \;|\; C \;|\; D}{1 \;|\; 0{,}4 \;|\; 0{,}9 \;|\; 0{,}8},$$

and the corrected set of non-dominated alternatives will be as follows:

$$\eta^{ND}_{cor} = \frac{A \;|\; B \;|\; C \;|\; D}{1 \;|\; 0{,}4 \;|\; 0{,}5 \;|\; 0{,}8}.$$

As we see, the greatest degree of non-dominance is the alternative A, so the choice of this particular project can be considered rational.

Then, when the greatest degree of non-domination has not one, but several alternatives, the DM can either choose by themselves one of them, based on some other additional considerations, or expand the circle of experts and solve the problem again as described above.

Conclusions

The theory of fuzzy sets is a mathematical apparatus which allows to describe the concepts that can not be clearly expressed. Applying this theory is appropriate in the cases where there is lack of information to make a decision or description of the situation by means of conventional sets makes a model "coarsened" that prevents achieving a satisfactory result.

The tasks of fuzzy mathematical modeling are generalization of conventional tasks of mathematical modeling. They are classified, depending on what elements are fuzzy, that results in the use of different approaches to thir solving. In particular, it can be an expansion on a plurality of sets, a reduction to a task that is not clearly defined, or to a multi-criteria optimization problem. The listed approaches belong to the indirect methods of solving tasks of FMP.

Fuzzy preferences can simulate situations in which information about the benefits of alternatives cannot be expressed unambiguously. They take these preferences "to some degree". In many cases, this makes it possible to build a more adequate mathematical model and simplify the task.

SELF-STUDY

Questions for assessment and self-assesment

1. Formulate the goal of achieving a fuzzy purpose.
2. What are the features of the Bellman-Zade approach to solving a task of achieving a fuzzy purpose?
3. How to take into account a goal and constraints in formulating and solving the achievement of clearly defined objectives?
4. Is it possible to formulate the task of achieving an ambiguous goal in the case when the goal and the constraint(s) represent a subset of different universal sets?
5. Formulate the general statement of the problem of fuzzy mathematical programming.
6. How fuzzy mathematical programming problems are classified?

7. What approaches are used to solve tasks of FMP?
8. What are the properties of the solution of the FMP problem?
9. What is the essence of the method of reduction to the task of achieving an ambiguous goal when solving the problems of the FMP?
10. What is the method of decomposition the set of levels when solving the problems of the FMP?
11. Expand the essence of the method of modal values in the application to the tasks of the FMP?
12. Describe the application of the method of reduction to the multicriterion problem in solving the problems of FMP.
13. What are the properties of solutions to the problem of FMP, based on the different approaches existing between them?
14. What is a fuzzy relation of preferences?
15. What properties does a fuzzy preference relation have? What does it characterize?
16. How can the preference relation be based on the relation of strict preference? What are its properties?
17. What relations are called λ-linear? Strongly linear? Describe their properties.
18. How do you make a rational choice of alternatives when you know the relation of preference to this set of alternatives?
19. Give a definition of an undominated alternative.
20. Define the concept of a clearly undominated alternative.
21. How is the rational choice of alternatives made, when you have some preference relation in the set of alternatives?
22. What types of convolution are used to select an alternative that is based on several relations of preferences? Outline the scope of their application.
23. How is rational choice of alternatives made, when it is given the relation of preference to the set of alternatives and the fuzzy preference in a set of properties?

Hands-on practice

Tasks A.

1. Solve the task of achieving a not clearly defined goal, according to the data presented in the table.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
G	0,8	0,6	0,7	0,4	0.2	0,1	0,1
C_1	0,5	0,3	0,5	0,4	0,6	0,5	0,4
C_2	0,9	0,8	0,7	0,6	0,5	0,4	0,3
C_3	0,3	0,5	0,7	1	0,9	0,6	0,2

2. The goal and constraints of the problem are described by the following membership functions:

$$\mu_C(x) = \begin{cases} -\frac{1}{4}(x-3)^2 + 1, & \text{if } x \notin (1;5) \\ 0 & \text{in other case,} \end{cases}$$

$$\mu_D(x) = \begin{cases} \frac{1}{6}x, & \text{if } x \in (0;6), \\ 1, & \text{if } x \geq 6, \\ 0 & \text{in other case.} \end{cases}$$

Solve the task of achieving a fuzzy defined goal.

3. Solve the task of achieving an fuzzy goal under the following conditions:

$$\mu_C(x) = \begin{cases} -(x-1)^2 + 1, & \text{if } x \notin (0;2), \\ 0 & \text{in other case,} \end{cases}$$

$$\mu_D(x) = \begin{cases} \frac{1}{2}|x|, & \text{if } x \notin (0;2), \\ 0 & \text{in other case.} \end{cases}$$

4. Solve the following problem of fuzzy mathematical programming:

$$\begin{aligned} & f_2(x_1, x_2) = x_1 - x_2 \to \min \\ & x_1 - 2x_2 \geq 2, \\ & 2x_1 + 5x_2 \leq 10, \\ & -12x_1 + 8x_2 \,\tilde{\leq}\, 24, \\ & x_1, x_2 \geq 0. \end{aligned}$$

5. Solve the following problem of fuzzy mathematical programming.

$$
\begin{aligned}
&f(x_1, x_2) = 2x_1 + 5x_2 \to \max \\
&x_1 + x_2 \geq 4, \\
&4x_1 + 6x_2 \mathrel{\tilde{\leq}} 24, \\
&3x_1 + 8x_2 \leq 24, \\
&x_1, x_2 \geq 0.
\end{aligned}
$$

6. Solve the following problem of fuzzy mathematical programming by the method of decomposition at the plural level.

$$
\begin{aligned}
&f_1(x_1, x_2) = 2x_1 + 3x_2 \to \min \\
&x_1 + 2x_2 \mathrel{\tilde{\geq}} 2, \\
&2x_1 + 5x_2 \leq 10, \\
&12x_1 + 8x_2 \leq 24, \\
&x_1, x_2 \geq 0.
\end{aligned}
$$

7. Solve the problem of fuzzy mathematical programming by the method of reduction to the multicriterion problem.

$$
\begin{aligned}
&f_1(x_1, x_2) = 2x_1 + 3x_2 \to \min \\
&x_1 + 2x_2 \geq 2, \\
&2x_1 + 5x_2 \leq 10, \\
&12x_1 + 8x_2 \leq 24, \\
&x_1, x_2 \geq 0.
\end{aligned}
$$

8. The set $X = \{x_1, x_2, x_3\}$, there are two fuzzy relations of preference given, the importance of which, respectively, $\lambda_1 = 0{,}7$, $\lambda_2 = 0{,}3$. Make a rational choice of an alternative from the set X according to the relations if

$$
R_1 = \begin{pmatrix} 1 & 0{,}5 & 0{,}45 \\ 0{,}3 & 1 & 0{,}2 \\ 1 & 1 & 0{,}4 \end{pmatrix}, \qquad R^2 = \begin{pmatrix} 1 & 0{,}5 & 0 \\ 0{,}8 & 1 & 0{,}1 \\ 1 & 0 & 1 \end{pmatrix}.
$$

9. At the given preferences, make a rational choice of the alternatives from the set: $X = \{x_1, x_2, x_3\}$.

$$R_1 = \begin{pmatrix} 1 & 0,5 & 0,45 \\ 0,3 & 1 & 0,2 \\ 0,1 & 0,5 & 1 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 1 & 0,8 & 0,4 \\ 0,3 & 1 & 0,2 \\ 1 & 1 & 1 \end{pmatrix}.$$

10. By the given relations the preference make a rational choice of the alternatives from the set: $X = \{x_1, x_2, x_3\}$, if $\lambda_1 = 0,4$, $\lambda_2 = 0,6$.

$$R_1 = \begin{pmatrix} 1 & 0,5 & 0,5 \\ 0,3 & 1 & 0,2 \\ 0,1 & 1 & 1 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 1 & 0,3 & 0,4 \\ 0,3 & 1 & 0,2 \\ 1 & 0,5 & 1 \end{pmatrix}.$$

Tasks B.

Using the theory of fuzzy sets, compile mathematical models of the following tasks.

1. Commercial Transport Company (distributor) buys goods of the same type from a group of Suppliers, and also transports it and sells it to Buyers. Suppose that *M* Suppliers and *N* Buyers are involved in co-operation. The constraints of each of them are known. Moreover, they are clearly described.

In addition, the distributor has information on:

- the purchase price per unit of product in each vendor $t_i, i = 1,2,...,M$;
- the sale price unit of finished product $s_j, i = 1,2,...,N$;
- the specific transport costs $c_{ij}, i = 1,2,...,M; j = 1,2,...,N$;
- the mandatory volume of proposals for the contract $p_i, i = 1,2,...,M$;
- the mandatory volume of demand for the contract $q_j, i = 1,2,...,N$;
- price of a unit of goods purchased outside the contract $k_i, i = 1,2,...,M$;
- the sale price of a unit of goods sold outside the contract $r_j, i = 1,2,...,N$;

It is necessary to determine what should be the volume of products x_{ij}, $i = 1,2,...,M$, $j = 1,2,...,N$,being purchased from each of the suppliers and sold to each of the buyers in order to minimize the cost of transportation of these goods and maximize the profit of the distributor.

2. An enterprise uses several production facilities. Each of them is characterized by the parameters that differ from other parameters (dimensions, lighting system used, atmospheric pollution level, surface reflectivity, lighting requirements and resource saving). Several types of luminaires with different technical parameters can be used to illuminate these premises.

The task is to rationally select the light sources for each room, taking into account the above characteristics, that is, what kind of lamp is necessary to apply to satisfy all the requirements the most.

3. At Zaporizhzhie iron ore plant, in the process of mining operations, hardening mixtures are used, consisting of viscous and inert materials. An inert filler in the preparation of such a mixture are wastes of energy, metallurgical and mining productions, in particular blast furnace slag (x_1), tailings of the central mining and processing plant (x_2), lime-dolomite materials (x_3), sand (x_4) and loam (x_5).

The task is to determine the composition of the mixture so that its cost is minimal and the strength corresponds to the standard conditions (should be 20-60 kg / cm2), the water should contain about 20% of the viscous constituents, and cement, calc-dolomite material and sand approximately 65, 9, 35 and 18% of the inert components in the mixture, respectively.

Consider that the dependence of the strength of the mixture of its components is described by such a function: $\varphi(x) = 467x_1 + 380x_2 - 54x_3 + 87x_4 - 120x_5 - 23{,}25$.

4. Several kinds of mixtures are used for the laying works at the mining enterprise, characterized by the following features: strength, cost, basicity, shrinkage, content of combustible components, porosity. Each of these features has a certain priority. It is necessary to choose the optimal mixture composition according to the set priorities.

5. A buyer chooses one of five models of washing machines. S/he evaluates each of them according to the following characteristics: cost, capacity, economy, overall dimensions, weight of loaded laundry.

Task: a) to formulate and make a decision, taking into account these conditions, the task of achieving a fuzzy defined goal;

b) formulate and solve the problem of choosing an alternative under fuzzy relations of preference. What assumptions should be made in each case?

6. A company management (CEO) should appoint one of three candidates for the post of the Chief Engineer. It is necessary to consider the following selection criteria: education, work experience, authority in the team, age, organizational abilities.

Tasks: a) to formulate and solve, taking into consideration these conditions as a task of achieving a fuzzy goal; b) articulate and solve the problem of making a choice using fuzzy relations of preference.

7. To make an alloy of lead, zinc and tin of a certain composition, raw materials are used in the form of five alloys of the same metals, but of different composition and different cost per 1 kg (see Table 5.1).

Determine how much alloy of each kind you need to take to make an alloy containing about 50% tin and about 25% zinc at an average cost.

Table 5.1

Type of alloy	Metal content, %			Unit cost, UAH / kg
	Plumbum	Zinc	Tin	
I	25	30	45	8
II	10	50	40	17
III	30	30	40	10
IV	40	25	35	12
V	10	70	20	15

8. At mine "Dobropilska", there are three operating extraction sites. The approximate content of sulfur, moisture and ash content of coal extracted on each of them is different (Table 5.2). It is known values of the maximum possible and the minimum required amount of coal production from each site, the amount of production costs for each site (Table 5.2). The planned output production at the mine is 3000 thousand tons. Considering the potential of each site, it is necessary to draw up a plan of mining operations in such a way that the extraction costs are minimal and all the requirements of the consumers to the quality of raw materials are fulfilled, in particular, the ash content should be approximately 47%, humidity - about 10%, sulfur content - about 3%.

Table 5.2

Characteristics of coal, % and the performance of the extraction site	Section number		
	1	2	3
Ash content	49	37	23
Humidity	7	8	10
Sulfur content	1,8	2,1	3
Costs, UAH	1184,210	1381,777	1083,515
Maximum volume of extraction, thousand tons	1650	1090	1270
Minimal amount of raw material extraction, thousand tons	1200	600	530

Tasks C

1 – 8. Solve the problems formulated in tasks *B* in one of the methods.

SECTION 6

MAKING DECISIONS IN RISKY AND UNCERTAIN CONDITIONS

By the end of this section you will:

- be aware of and know methods of making decisions in risky and uncertain conditions and how to apply them when making a decision;
- have practiced solving problems and tasks using the described methods.

6.1 Concept of situation of decision-making

Let us now consider the situation when the quality of the solution depends on the external factors, which the DM (or the management body) does not influence. We will also assume that these parameters and perturbations remain unchanged in time, i.e. the model is static.

The static decision-making model, which is based on the game-theoretic concept, is well known and common for many real-world circumstances of one-off choice of options (plans, actions, alternatives, strategies etc.) associated with the uncertain influence of the environment on the choice situation that decision - making body holds.

Investigating the static decision-making models, we will proceed from a scheme in which the following assumptions are provided:

1) the management body has a set of mutually exclusive decisions: $\Phi = \{\varphi_1, \varphi_2, ...\varphi_m\}$, one of which must be chosen;

2) the environment C is described by a set of mutually exclusive states: $\Theta = \{\theta_1, \theta_2, ...\theta_n\}$, and can be in one of them, however at the moment of decision-making by the management body it is not known which state exactly it is in or will be;

3) an estimating functional is defined: $F = \{f_{j,k}\}$ characterizing the "benefit" or "loss" of the management body when choosing a solution $\varphi_k \in \Phi$, if the medium will be or is in the state $\theta_j \in \Theta$.

Based on these assumptions, the decision-making process under uncertainty can be described by the following scheme:

1) Forming a set of possible decisions of the management body: $\Phi = \{\varphi_1, \varphi_2, ...\varphi_m\}$, and sets of states of the environment: $\Theta = \{\theta_1, \theta_2, ...\theta_n\}$

2) Determining and setting of the main indicators of efficiency and utility, which are included in calculation of the estimated functional: $F = \{f_{j,k}\}$;

3) Determining the information situation that describes the environmental behaviour strategy C by the management body;

4) Selecting criteria for decision making in a set of criteria that characterize the information management situation determined by the management body.

5) Adopting or correcting the optimal decision according to the chosen criterion.

Let's formulate the necessary definitions.

The *situation of decision-making* is referred to as three elements: $\{\Phi, \Theta, F\}$, in which:

$\Phi = \{\varphi_1, \varphi_2, \dots \varphi_m\}$ is a set of possible decisions of the management body;

$\Theta = \{\theta_1, \theta_2, \dots \theta_n\}$ – a set of possible states of the environment;

$F = \{f_{j,k}\}$ is an estimated functionality, where is $f_{j,k} = f(\theta_j, \varphi_k)$.

In an expanded form, the decision-making situation is characterized by the following matrix:

$$\begin{array}{ccccccc}
 & \varphi_1 & \varphi_2 & \cdots & \varphi_k & \cdots & \varphi_m \\
\theta_1 & f_{11} & f_{12} & \cdots & f_{1k} & \cdots & f_{1m} \\
\theta_2 & f_{21} & f_{22} & \cdots & f_{2k} & \cdots & f_{2m} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\theta_j & f_{j1} & f_{j2} & \cdots & f_{jk} & \cdots & f_{jm} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\theta_n & f_{n1} & f_{n2} & \cdots & f_{nk} & \cdots & f_{nm}
\end{array}$$

The category of evaluative functional is closely related to such concepts as efficiency, utility, loss, risk and under. At the same time, the choice of this or that form of a functional depends on the specific management tasks. Usually two of its forms are used: those that determine the utility and/or those that determine the loss.

If an estimated functional determines the efficiency, utility, profit etc., i.e. when making a decision the management body proceeds from the need to reach their maximum, then it is said that it (the functional) has a positive ingredient. In this case, the estimated functional is indicated in the following way: $F = F^+ = \{f^+_{j,k}\}$.

When the management body proceeds from the need to achieve a minimum of evaluation functional (i.e, it shows losses, risk), it means that it has a negative ingredient. This fact is written as follows: $F = F^- = \{f^-_{j,k}\}$.

An informational situation of decision-making situation is referred to a degree of uncertainty gradation in the choice of states by the environment itself from a given set of "Θ" at the moment of decision-making by the management body.

The following informational situations are distinguished [35]:

I_1 - when the distribution of a priori probabilities on the elements of the set of states of the environment Θ is given, this situation is also called the situation of decision-making under risky conditions;

I_2 – there probability distribution with unknown parameters is given;

I_3 - there are systems of linear relations of orders on the components of the a priori distribution of the states of the medium *C;*

I_4 - when the distribution of probabilities in the set of states of the environment Θ is unknown;

I_5 – the antagonistic interests of the environment in the decision-making process*;*

I_6 - "intermediate" choice of the environment of their states between I_1 *and* I_5 *;*

I_7 –the existence of a fuzzy set of states of the environment.

A decision making criterion is referred to the algorithm which is defined for each of decision-making situations and the informational situation *I*, that allows to choose the only one optimal solution φ_0 from the set F or to make a set of such solutions, which are called *equivalent against given criterion.*

In each informational situation "*I*" several criteria can be applied. The choice of a particular one is performed by the decision-making body.

6.2 Criteria for decision-making under risky conditions

The first informational situation I_1 is characterized by a given distribution of a priori probabilities on elements of the set Θ , namely*:* $p = (p_1, p_2 \ldots p_n)$, where $p_j = p(\theta = \theta_j), \sum_{j=1}^{n} p_j = 1$. It is widely spread when simulating practical decision-making tasks for risky conditions, since it allows us to use effectively the constructive methods of the theory of probability in the process of developing a whole scientific direction - the theory of statistical decisions.

Note that in real tasks or problems, the calculation of the a priori distribution*:* $p = (p_1, p_2 \ldots p_n)$, is carried out either by processing a large amount of statistical material or on the basis of analytical methods based on hypotheses on the behaviour of the environment and the application of the methods and theorems of the probability theory. Both methods give results which are approximate to some extent, since certain difficulties and constraints (they relate to cost, time, under) arise from the processing of statistical data. When it comes to the use of analytical methods, it is necessary to make certain assumptions, sometimes at the expense of the accuracy of the process description. The resulting a priori distribution of probabilities is called *objective*. However, sometimes the use of such methods is impossible, since there is not enough statistical material, the environment is characterized by a complex "behaviour" and, as a result of this application of analytical methods, requires additional research, which leads to significant costs and time. In these circumstances, the decision-making body can use the values of the a priori distribution of the probability of thought and representation of experienced experts who are well-orientated in the situation to formulate estimates. Such a definition of probability is called *subjective*.

Let's describe the criteria for making decisions in situation I_1.

1. ***Bayesian criterion*** **(*average*).** The meaning of this criterion is to maximize the mathematical expectation of the estimated functional.

According to Bayesian criterion, the optimal solutions of B (or a set of such solutions) are those for which the mathematical expectation of the estimated functional acquires the largest (or the smalles) possible value, namely:

$$\varphi_{k_0} : B^+(\varphi_{k_0}, p) = \max_{\phi_k \in \Phi} B^+(\varphi_k, p), \quad k = \overline{1,m}, \quad B^+(\varphi_k, p) = \sum_{i=1}^{n} p_i f_{ik}^+ \tag{6.1}$$

for a functional with a positive ingredient;

$$\varphi_{k_0} : B^-(\varphi_{k_0}, p) = \min_{\phi_k \in \Phi} B^-(\varphi_k, p), \quad k = \overline{1,m}, \quad B^-(\varphi_k, p) = \sum_{i=1}^{n} p_i f_{ik}^- \tag{6.2}$$

for a functional with a negative ingredient.

The Bayesian criterion is most used in the informational situation I_1. It is expedient to use it when the situation repeats many times, since in these conditions, the average benefit value is maximized (or the average risk is minimized).

2. ***The criterion for the minimum dispersion of the estimated functional.*** For each solution $\varphi_k \in \overline{\Phi}$ we define the average value $B^+(\varphi_k, p)$ of the estimated functional and dispersion σ_k^2 in the following form:

$$B^+(\varphi_k, p) = \sum_{i=1}^{n} p_i f_{ik}^+ , \tag{6.3}$$

$$\sigma_k^2 = \sum_{i=1}^{n} (f_{ik}^+ - B^+(\varphi_k, p))^2 \, p_i . \tag{6.4}$$

The dispersion describes the scattering of random values of the estimated functional for the solution φ_k relative to its average value $B^+(\varphi_k, p)$.

The essence of the criterion for minimizing the dispersion of the estimated functional is to find a solution $\varphi_{k_0} \in \Phi$ (or a set of solutions $\overline{\Phi}$) for which the following equality is true:

$$\sigma^2\left(p, \varphi_{k_0}\right) = \min_{\varphi_k \in \Phi} \sigma_k^2\left(p, \varphi_k\right). \qquad (6.5)$$

The main disadvantage of this criterion is that the dispersion in the decision $\varphi_{k_1} \in \Phi$ may be less than that in the the decision $\varphi_{k_2} \in \Phi$, while $B^+(\varphi_{k_1}, p) < B^+(\varphi_{k_2}, p)$. In other words, the criterion for the minimum dispersion, on the one hand, is auxiliary, and on the other, its adoption requires additional determination and a slight change in the form of the dispersion σ_k^2, by one of the following methods, for example:

$$\sigma^2(\varphi_k) = \sum_{i=1}^{n} (f_{ik}^+ - \frac{1}{m} \sum_{i=1}^{m} B^+(\varphi_k, p))^2 p_i \qquad (6.6)$$

or

$$\sigma^2(\varphi_k) = \sum_{i=1}^{n} (f_{ik}^+ - \max_{\varphi_s \in \Phi} B^+(\varphi_s, p))^2 p_i. \qquad (6.7)$$

If the estimated functional is given in the form of a negative ingredient, namely: $F = F^- = \left\{f_{j,k}^-\right\}$, then the solution $\varphi_{k_0} \in \Phi$ can also be found using condition (6.5), but here the value σ_k^2 is determined by one of the following methods:

$$\sigma_k^2 = \sum_{i=1}^{n} (f_{ik}^- - B^-(\varphi_k, p))^2 p_i, \qquad (6.8)$$

$$\sigma^2(\varphi_k) = \sum_{i=1}^{n} (f_{ik}^- - \frac{1}{m} \sum_{i=1}^{m} B^-(\varphi_k, p))^2 p_i \qquad (6.9)$$

or

$$\sigma^2(\varphi_k) = \sum_{i=1}^{n} (f_{ik}^- - \min_{\varphi_s \in \Phi} B^-(\varphi_s, p))^2 p_i, \qquad (6.10)$$

Moreover, $B^-(\varphi_k, p) = \sum_{i=1}^{n} p_i f_{ik}^-$.

3. ***The criterion for maximizing probability of the distribution of the estimated functional.*** We fix the value α that satisfies the following condition: $\alpha_1 \leq \alpha \leq \alpha_2$, and $\alpha_1 = \min_j \min_k f_{jk}^+$, $\alpha_2 = \max_j \max_k f_{jk}^+$.

For each $\varphi_k \in \Phi$ solution, probability of $p\left(f_{jk}^+ \geq \alpha\right)$ is determined by the fact that the value of the estimated functional will be not less than α, when the environment is in the state θ_j and thesolution φ_k is chosen.

The meaning of the criterion for maximizing the probability of the distribution of the estimated functional is to determine the solution $\varphi_{k_0} \in \Phi$ (or the set of solutions $\overline{\Phi}$) for which this probability will be maximal, that is,

$$\varphi_{k_0} : p(f_{jk_0}^+ \geq \alpha) = \max_{\varphi_k \in \Phi} p(f_{jk}^+ \geq \alpha). \tag{6.11}$$

When using this criterion, the management body proceeds from the need to adopt a specific level of the value of α, and those solutions, for which the condition (6.11) is satisfied, are considered optimal.

When probability values α and solution φ_k are fixed, the inequality: $f_{ik}^+ \geq \alpha$ defines the set of states of the environment $\theta_{\alpha k}$, and probability of $p(f_{jk}^+ \geq \alpha)$ is calculated as follows:

$$p(f_{jk}^+ \geq \alpha) = \sum_{\theta_j \in \theta_{\alpha k}} p(f_{jk}^+ \geq \alpha), \quad \theta_{\alpha k} = \{\theta_j : f_{jk}{}^+ \geq \alpha\}. \tag{6.12}$$

If the estimated functional has a negative ingredient F^-, then it is necessary to determine the probability $p(f_{ik}^- \leq \alpha)$, and the criterion becomes the following:

$$p(f_{jk_0}^- \leq \alpha) = \max_{\varphi_k \in \Phi} p(f_{jk}^- \leq \alpha), \tag{6.13}$$

where

$$p(f_{jk}^- \leq \alpha) = \sum_{\theta_j \in \theta_{\alpha k}} p(f_{jk}^- \leq \alpha), \quad \theta_{\alpha k} = \{\theta_j : f_{jk}{}^- \leq \alpha\}.$$

4. ***Modal criterion.*** The idea of this criterion is that the decision-making body comes from the most probable state of the environment. Assume that there is a single value j^* that ensures the fulfillment of this condition:

$$p(\theta_{j^*}) = \max_{\theta_j \in \Theta} p(\theta j). \tag{6.14}$$

Then, the management body believes that the environment is in the state of θ_{j*} and chooses the solution φ_0, for which $f_{j0}^{+} = \max_{k} f_{j*k}^{+}$, when the functional has a positive ingredient or $f_{j0}^{-} = \min_{k} f_{j*k}^{-}$, when the functional is characterized by a negative ingredient.

Note that here the situation is possible when the maximum probability value is reached simultaneously in several elements of the set Θ, that is:

$$p(\theta_{j_1}) = p(\theta_{j_2}) = \dots p(\theta_{j_s}) = \max_{\theta_j \in \Theta} p(\theta j),$$

then the optimal solution must be chosen based on the following condition:

$$\frac{1}{s}\sum_{r=1}^{s} f_{j_r 0}^{+} = \max_{\varphi_k \in \Phi} \frac{1}{s}\sum_{r=1}^{s} f_{j_r k}^{+}. \qquad (6.15)$$

The advantage of the modal criterion is its simplicity. Firstly, it is enough only to identify the most probable states of the environment, and besides, there is no need even to know the numerical values of these probabilities. Secondly, the calculation of the values of the estimated functional can be performed only for the most probable states, which significantly increases the speed of decision-making.

Among the disadvantages and shortcomings of the criterion, it should be mentioned the possibility of the solution optimal for the modal criterion is not always having the highest Bayesian value.

5. *Combined criterion.* It is a combination of the Bayesian criteria and a dispersion minimum which takes into account the natural desire of the management body to provide the best average (Bayesian criterion) and minimal dispersion.

Let's choose the value λ, $0 \le \lambda \le 1$ and for each of the solutions φ_k, $k = 1, 2, \dots m$, we calculate the value of the criterion by the following formula:

$$k(\varphi_k, p) = (1-\lambda)\left(B^{+}(\varphi_k, p)\right)^2 - \lambda\sigma^2(\varphi_k, p), \qquad (6.16)$$

The best solution is φ_0, for which the following condition is fulfilled:

$$k(\varphi_0, p) = \max_{\varphi_k \in \Phi} k(\varphi_k, p).$$

Note, that at the same time, the value of the coefficient λ is established on the basis of which particular criterion (Bayesian or minimum dispersion) is to be given a greater advantage. If $\lambda = 0$, then the criterion coincides with the Bayesian criterion $k(\varphi_k, p)$, and when $\lambda = 1$ – with the criterion of the minimum dispersion.

Let's consider two values:

$$\lambda^* = \min_{\varphi_k \in \Phi} \frac{\left[\sum_{j=1}^{n} p_j f_{jk}^+\right]^2}{\sum_{j=1}^{n} p_j (f_{jk}^+)^2}; \qquad \lambda^{**} = \max_{\varphi_k \in \Phi} \frac{\left[\sum_{j=1}^{n} p_j f_{jk}^+\right]^2}{\sum_{j=1}^{n} p_j (f_{jk}^+)^2}. \tag{6.17}$$

Obviously, they satisfy such inequality: $0 \le \lambda^* \le \lambda^{**} \le 1$. Here are the statements which are given below.

Lemma 6.1 If parameter λ satisfies such a condition $0 \le \lambda \le \lambda^*$, then $k(\varphi_k, p) \ge 0$ for any solution $\varphi_k \in \Phi$.

Lemma 6.2. When parameter λ satisfies such a condition $\lambda^{**} \le \lambda \le 1$, then $k(\varphi_k, p) \le 0$ for every decision $\varphi_k \in \Phi$.

The proof of these statements is given in the monograph [35].

Thus, we can conclude that when $0 \le \lambda \le \lambda^*$, then in the combined criterion the advantage is given to the Bayesian criterion in comparison with the minimum dispersion criterion, and when $\lambda^{**} \le \lambda \le 1$, then the minimum criterion for dispersion is more taken into account.

6. ***The criterion of the minimal entropy of a mathematical expectation of an estimated functional.*** Assume, that $f_{jk}^+ > 0$ for all values of $j = \overline{1, n}$ and $k = \overline{1, m}$. For each of the possible solutions of $\varphi_k \in \Phi$ we calculate the entropy of the mathematical expectation of the estimated functional by the following formula*:*

$$H(p, \varphi_k) = -\sum_{i=1}^{n} \left(\frac{p_i f_{ik}^+}{\sum_{j=1}^{n} p_j f_{jk}^+} \right) \cdot \ln \left(\frac{p_i f_{ik}^+}{\sum_{j=1}^{n} p_j f_{jk}^+} \right). \tag{6.18}$$

The solution φ_0 is considered optimal when it has a minimum entropy, that is:

$$\varphi_0 : H(p, \varphi_0) = \min_k H(p, \varphi_k).$$

When the condition: $f_{jk}^+ > 0$, $j = \overline{1, n}$ and $k = \overline{1, m}$, is not executed, then you can go to the category of losses by taking advantage of the transformation*:* $\tilde{f}_{jk}^- = \max_{\substack{\varphi_k \in \Phi \\ \theta_j \in \Theta}} f_{jk}^+ - f_{jk}^+$ and find the solution φ_0 on the criterion of the minimum entropy of the mathematical expectation of the estimated functional $H(p,\varphi_k)$ when $\varphi_k \in \Phi$, where:

$$H(p,\varphi_k) = -\sum_{i=1}^{n} \left\{ \left(\frac{p_i f_{ik}^-}{\sum_{j=1}^{n} p_j f_{jk}^-} \right) \cdot \ln \left(\frac{p_i f_{ik}^-}{\sum_{j=1}^{n} p_j f_{jk}^-} \right) \right\}, \quad f_{jk}^- > 0. \tag{6.19}$$

7. ***Conditional decisions.*** Assume that in the decision-making situation I we can apply criteria for some set: $K = \{k_1, k_2, \ldots, k_n\}$. One of them is considered to be the main one, and for the other criteria the restrictions are given in the following form: $c_i \le k_i \le c_i, i = 1,n$.

In this case, the decision taken by the management body on the main criterion is called *conditional.*

Here it is worth taking into account the following considerations:

1) since the search for optimal solutions under the conditions described is reduced to a search of options, then constraints in the form of equalities are not always appropriate to use since they can lead to the absence of solutions (the problem will not have any feasible solution at all);

2) a conditional solution can be chosen without the main criterion, then it will represent a solution of such a system of inequalities:

$$c_i \le k_i \le c_i, i = 1,n.$$

Let's consider an example of applying making-decision criteria under risky conditions, using the example of the problem presented below.

Example 6.1. The team of power network installers consists of 5 workers, who perform repair and re-installation works in emergency situations. Management needs to make a decision on changing the number of workers. The following options are possible:

φ_1 – do not change the number of workers;

φ_2 – increase the number of workers by combining shifts (the team will be a variable, i.e. team-members may be different in different shifts);

φ_3 – increase the number of workers;

φ_4 – reduce the number of workers.

It is necessary to take into account the following possible situations:

θ_1 – the average number of accidents during the day will be high (significantly increase);

θ_2 – the average number of accidents during the day will be moderately high (insignificantly increase);

θ_3 – there will be a few accidents per day (their number will not change);

θ_4 – during the day there will be few accidents (their number will decrease);

θ_5 – there will be no accidents and/or emergent situations.

According to experts' estimates, probability of each of the situations and effectiveness of the decisions taken in them can be described as following:

p		φ_1	φ_2	φ_3	φ_4
0,2	θ_1	3	4	7	0
0,4	θ_2	5	5	6	1
0,2	θ_3	8	2	2	6
0,1	θ_4	3	1	0	7
0,1	θ_5	1	0	0	9

It is necessary to choose optimal solution.

Solve the problem using the criteria of the first informational situation.

The Bayes criterion. Since in our problem we use a functional with a positive ingredient to evaluate possible solutions (it describes the effectiveness of solutions), we will use formulas (6.1).

We calculate Bayes value for each of the solutions, namely:

$$B^{+}(\varphi_1, p) = \sum_{i=1}^{5} f_{i1}p_i = 3\cdot 0{,}2 + 5\cdot 0{,}4 + 8\cdot 0{,}2 + 3\cdot 0{,}1 + 1\cdot 0{,}1 = 4{,}6.$$

$$B^{+}(\varphi_2, p) = \sum_{i=1}^{5} f_{i2}p_i = 4\cdot 0{,}2 + 5\cdot 0{,}4 + 2\cdot 0{,}2 + 1\cdot 0{,}1 + 0\cdot 0{,}1 = 3{,}3.$$

$$B^{+}(\varphi_3, p) = \sum_{i=1}^{5} f_{i1}p_i = 7\cdot 0{,}2 + 6\cdot 0{,}4 + 2\cdot 0{,}2 + 0\cdot 0{,}1 + 0\cdot 0{,}1 = 4{,}2.$$

$$B^{+}(\varphi_4, p) = \sum_{i=1}^{5} f_{i1}p_i = 0\cdot 0{,}2 + 1\cdot 0{,}4 + 6\cdot 0.2 + 7\cdot 0{,}1 + 9\cdot 0{,}1 = 3{,}2.$$

It is easy to notice that the highest Bayes value is the solution φ_1: $B(\varphi_1, p) = 4{,}6$, so its choice can be considered rational. Therefore, it is necessary to leave the number of workers unchanged.

The criterion for minimizing dispersion of the estimated functional. By formula (6.4), we calculate the value of dispersion for each of the possible solutions, that is,

$$\sigma^2(\varphi_1, p) = \sum_{i=1}^{5} (f_{i1} - 4{,}6)^2 p_i = 2{,}56\cdot 0{,}2 + 0{,}16\cdot 0{,}4 + 11{,}56\cdot 0{,}2 + 2{,}56\cdot 0{,}1 +$$
$$+ 12{,}96\cdot 0{,}1 = 4{,}44.$$

$$\sigma^2(\varphi_2, p) = \sum_{i=1}^{5}(f_{i2} - 3{,}3)^2 p_i = 0{,}49 \cdot 0{,}2 + 2{,}89 \cdot 0{,}4 + 1{,}69 \cdot 0{,}2 + 5{,}29 \cdot 0{,}1 +$$
$$+10{,}89 \cdot 0{,}1 = 3{,}21.$$

$$\sigma^2(\varphi_3, p) = \sum_{i=1}^{5}(f_{i3} - 4{,}2)^2 p_i = 7{,}84 \cdot 0{,}2 + 3{,}24 \cdot 0{,}4 + 4{,}84 \cdot 0{,}2 + 17{,}64 \cdot 0{,}1 +$$
$$+17{,}64 \cdot 0{,}1 = 7{,}36.$$

$$\sigma^2(\varphi_4, p) = \sum_{i=1}^{5}(f_{i4} - 3{,}2)^2 p_i = 10{,}24 \cdot 0{,}2 + 4{,}84 \cdot 0{,}4 + 7{,}84 \cdot 0{,}2 + 14{,}44 \cdot 0{,}1 +$$
$$+33{,}64 \cdot 0{,}1 = 10{,}36.$$

The lowest value of the dispersion has a solution φ_2 that involves increasing the number of workers by combining shifts, so its choice on this criterion will be rational.

Let's consider the modifications of this method.

The first one is described by the formula (6.6). Firstly, we calculate the average Bayes value for all the criteria, namely:

$$\frac{1}{m}\sum_{j=1}^{m} B^{+}(\varphi_j, p) = 3{,}825.$$

Now we calculate the value of dispersion (the variances) with respect to it, that is,

$$\sigma_1^2(\varphi_1, p) = \sum_{i=1}^{5}(f_{i1} - 3{,}825)^2 p_i = 0{,}68 \cdot 0{,}2 + 1{,}38 \cdot 0{,}4 + 17{,}43 \cdot 0{,}2 + 0{,}68 \cdot 0{,}1 +$$
$$+7{,}98 \cdot 0{,}1 = 5{,}04.$$

$$\sigma_2^2(\varphi_2, p) = \sum_{i=1}^{5}(f_{i2} - 3{,}85)^2 p_i = 0{,}03 \cdot 0{,}2 + 1{,}38 \cdot 0{,}4 + 3{,}33 \cdot 0{,}2 + 7{,}98 \cdot 0{,}1 +$$
$$+14{,}63 \cdot 0{,}1 = 3{,}48.$$

$$\sigma_3^2(\varphi_3, p) = \sum_{i=1}^{5}(f_{i3} - 3{,}85)^2 p_i = 10{,}08 \cdot 0{,}2 + 4{,}73 \cdot 0{,}4 + 3{,}33 \cdot 0{,}2 + 14{,}63 \cdot 0{,}1 +$$
$$+14{,}63 \cdot 0{,}1 = 7{,}5.$$

$$\sigma_4^2(\varphi_4, p) = \sum_{i=1}^{5}(f_{i4} - 3{,}85)^2 p_i = 14{,}63 \cdot 0{,}2 + 7{,}98 \cdot 0{,}4 + 4{,}73 \cdot 0{,}2 + 10{,}08 \cdot 0{,}1 +$$
$$+26{,}78 \cdot 0{,}1 = 10{,}75.$$

Obviously, the minimum value of dispersion is obtained for the solution φ_2, so the choice of this alternative can be considered rational.

The second modification is described by the formula (6.7). The maximum Bayes value is $\max_j B^+(\varphi_j, p) = 4,6$. We calculate deviations of the values of the estimating functional for each of the solutions from it, namely:

$$\sigma_1^2(\varphi_1, p) = \sum_{i=1}^{5} (f_{i1} - 4,6)^2 p_i = 2,56 \cdot 0,2 + 0,16 \cdot 0,4 + 11,56 \cdot 0,2 + 2,56 \cdot 0,1 +$$
$$+12,96 \cdot 0,1 = 4,44.$$

$$\sigma_2^2(\varphi_2, p) = \sum_{i=1}^{5} (f_{i2} - 4,6)^2 p_i = 0,36 \cdot 0,2 + 0,16 \cdot 0,4 + 6,76 \cdot 0,2 + 12,96 \cdot 0,1 +$$
$$+21,16 \cdot 0,1 = 4,9.$$

$$\sigma_3^2(\varphi_3, p) = \sum_{i=1}^{5} (f_{i3} - 4,6)^2 p_i = 5,76 \cdot 0,2 + 1,96 \cdot 0,4 + 6,76 \cdot 0,2 + 21,16 \cdot 0,1 +$$
$$+21,16 \cdot 0,1 = 7,52.$$

$$\sigma_4^2(\varphi_4, p) = \sum_{i=1}^{5} (f_{i4} - 4,6)^2 p_i = 21,16 \cdot 0,2 + 12,96 \cdot 0,4 + 1,96 \cdot 0,2 + 5,76 \cdot 0,1 +$$
$$+19,36 \cdot 0,1 = 12,32.$$

The minimum value of the modified dispersion corresponds to the solution φ_1, that is why its choice can be considered rational.

Now let's solve the problem using the *criterion of maximizing distribution probability* of the estimated functional. We compute the necessary parameters: $\alpha_1 = \min_i \min_j f_{ij}$ *and* $\alpha_2 = \max_i \max_j f_{ij}$, according to our data $\alpha_1 = 0$ i $\alpha_2 = 9$.

Now let’s choose the number *α* that satisfies the following condition: $\alpha_1 \le \alpha \le \alpha_2$, assume that *α* = 5. By the formulas (6.12) we calculate the probability $p(f_k^+ \ge \alpha)$ for each solution φ_k, that is:

φ_1: $p(f_1^+ \ge 5) = 0,4 + 0,2 = 0,6$;

φ_2: $p(f_2^+ \ge 5) = 0,4$;

φ_3: $p(f_3^+ \ge 5) = 0,2 + 0,4 = 0,6$;

φ_4: $p(f_4^+ \ge 5) = 0,2 + 0,1 + 0,1 = 0,4$.

The maximum value of probability has two alternatives: φ_1 and φ_3, that is, rational by this criterion, the following decisions can be considered: not to change the number of workers or to increase it.

Now we apply the *modal criterion* for the solution of the problem.

With this purpose, we determine the state of the environment, which has the greatest probability. For the formulated problem, this will be the state *θ2,* that is, the average number of accidents during the day will be moderately higher (insignificantly increase). Estimates of solutions for this state have the following meanings: $f_{21}^{+}=5$, $f_{22}^{+}=5$, $f_{23}^{+}=6$, $f_{24}^{+}=1$.

The highest value corresponds to the decision φ_3. Therefore, its choice can be considered rational, that is, it is necessary to increase the number of workers.

Let's consider some more criteria that can be used in this situation.

The *criterion for the minimal entropy of the mathematical expectation of the estimated functional.* For each of the possible solutions, we calculate the entropy of the mathematical expectation of the estimated functional by the formula (6.18).

In our case:

$$H(p,\varphi_1)=-\sum_{i=1}^{n}\left\{\left(\frac{p_i f_{i1}^{+}}{\sum_{j=1}^{n} p_j f_{j1}^{+}}\right)\cdot\ln\left(\frac{p_i f_{i1}^{+}}{\sum_{j=1}^{n} p_j f_{j1}^{+}}\right)\right\}=1{,}1731.$$

$$H(p,\varphi_2)=-\sum_{i=1}^{n}\left\{\left(\frac{p_i f_{i2}^{+}}{\sum_{j=1}^{n} p_j f_{j2}^{+}}\right)\cdot\ln\left(\frac{p_i f_{i2}^{+}}{\sum_{j=1}^{n} p_j f_{j2}^{+}}\right)\right\}=1{,}008.$$

$$H(p,\varphi_3)=-\sum_{i=1}^{n}\left\{\left(\frac{p_i f_{i3}^{+}}{\sum_{j=1}^{n} p_j f_{j3}^{+}}\right)\cdot\ln\left(\frac{p_i f_{i3}^{+}}{\sum_{j=1}^{n} p_j f_{j3}^{+}}\right)\right\}=0{,}909.$$

$$H(p,\varphi_4)=-\sum_{i=1}^{n}\left\{\left(\frac{p_i f_{i4}^{+}}{\sum_{j=1}^{n} p_j f_{j4}^{+}}\right)\cdot\ln\left(\frac{p_i f_{i4}^{+}}{\sum_{j=1}^{n} p_j f_{j4}^{+}}\right)\right\}=0{,}96.$$

The criterion has the minimum value for the solution φ_3. Thus, the choice of alternative φ_3 – to increase the number of workers, will be optimal.

Combined criterion. Consider the problem of choosing a solution for this criterion, using different values of the parameter λ. We calculate the boundary values of the parameter, which determine the advantage of the criteria by the formulas (6.17), namely:

$$\lambda^* = \min_k \frac{\left[\sum_{j=1}^{n} p_j f_{jk}^+\right]^2}{\sum_{j=1}^{n} p_j (f_{jk}^+)^2} = 0,497; \qquad \lambda^{**} = \max_k \frac{\left[\sum_{j=1}^{n} p_j f_{jk}^+\right]^2}{\sum_{j=1}^{n} p_j (f_{jk}^+)^2} = 0,827.$$

Now we calculate the value of the criterion, when $\lambda = 0,4$ $(\lambda < \lambda^*)$, that is

$$k(p,\varphi_1) = 4,6^2(1-0,4) - 0,4 \cdot 4,44 = 10,92.$$

$$k(p,\varphi_2) = 3,3^2(1-0,4) - 0,4 \cdot 3,21 = 5,25.$$

$$k(p,\varphi_3) = 4,3^2(1-0,4) - 0,4 \cdot 6,61 = 7,64.$$

$$k(p,\varphi_4) = 3,2^2(1-0,4) - 0,4 \cdot 10,36 = 2.$$

The maximum value of the criterion corresponds to the decision φ_1, that is, when $\lambda < \lambda^*$, the choice of this solution will be rational.

Consider the problem of choice in the condition provided that $\lambda = 0,9$ ($\lambda > \lambda^{**}$), namely:

$$k(p,\varphi_1) = -1,88; \quad k(p,\varphi_2) = -1,8; \quad k(p,\varphi_3) = -4,86; \quad k(p,\varphi_4) = -8,3.$$

The maximum value of the criterion corresponds to the alternative φ_2, that is, if $\lambda > \lambda^*$, the choice of this solution will be rational.

In the case, when $\lambda = 0,662$ $(\lambda^* < \lambda < \lambda^{**})$, the criterion acquires the following values:

$$k(p,\varphi_1) = 4,2128; \; k(p,\varphi_2) = 1,5558; \; k(p,\varphi_3) = 1,09; \; k(p,\varphi_4) = -3,3972.$$

So, the solution φ_1 will be optimal.

Let's put the results calculated using different criteria in a table (Table. 6.1).

Table 6.1

Results of calculations and decision making according to the different criteria in risky conditions

Criteria name	The value of the criterion for decisions				Optimal solution
	φ_1	φ_2	φ_3	φ_4	
Bayesian	4,6	3,3	4,2	3,2	φ_1
Minimum dispersion	4,44	3,21	7,36	10,36	φ_2
Modification 1	5,04	3,48	7,5	10,75	φ_2
Modification 2	4,44	4,9	7,52	12,32	φ_1
Maximizing the probability distribution of the estimated functional	0,6	0,4	0,6	0,4	φ_1, φ_2
Modal	5	5	6	1	φ_3
Minimum entropy	1,1731	1,008	0,909	0,96	φ_3
Combined, λ = 0,4	10,92	5,25	7,64	2	φ_1
Combined, λ = 0,9	–1,88	–1,8	–4,86	–8,3	φ_2
Combined, λ = 0,662	4,2128	1,5558	1,09	–3,3972	φ_1

Analyzing the results, we can draw a conclusion, that the use of different criteria gives different optimal alternatives, and therefore, before making a decision it is necessary to determine which criterion it will be chosen on, taking into consideration the terms of the task and the requirements for the decision. For example, in the considered problem, the alternative φ_4 will not be optimal for any of the criteria, the alternative φ_3 will be optimal only by the modal criterion, that is, a decision maker must choose between two alternatives: φ_1 - not to change the number of workers and φ_2 – to increase the number of workers by combining shifts (the team will have a variable members).

6.3 Bayesian sets

The Bayesian criterion makes it possible in the information situation to investigate the synthesis problem while choosing the optimal solution according to the probability distribution: $p = (p_1, ..., p_n)$, *i*n the set of environment states *C*. We denote by Δ the set of possible values of the a priori probability distribution vector, namely:

$$\Delta = \left\{ (p_1, \ldots, p_n) : 0 \le p_j \le 1 \, , \, j = 1, \ldots n \, , \, \sum_{i=1}^{n} p_j = 1 \right\},$$

and consider $(n - 1)$ - dimensional simplex:

$$P_{n-1} = \left\{ (p_1, \ldots, \, p_{n-1}) : 0 \le p_j \le 1, \, j = 1, \ldots n - 1, \, \sum_{i=1}^{n-1} p_j \le 1 \right\}. \tag{6.20}$$

It is a projection of a flat set Δ, in $(n - 1)$ - dimensional space of values of the first $(n - 1)$ components of the vector of the a priori distribution $p = (p_1, ..., p_n)$.

The essence of the synthesis problem is to partition the simplex P_{n-1} into sets, $S_{\varphi_k} \subset P_{n-1}, k = 1, \ldots m$, satisfying the following conditions:

1) $S_{\varphi_i} \cap S_{\varphi_k} = \varnothing$,when $i \ne k$;

2) $\bigcup_{k=1}^{m} S_{\varphi_k} = P_{n-1}$.

3) The solution $\varphi_k \in \Phi$ will be optimal for the Bayesian criterion, if $p \subset S_{\varphi_k}$.

We will call the set S_{φ_k} as *the Bayesian set* of a priori probabilities: $p = (p_1, ..., p_n)$ in respect to solution φ_k, the solution itself $\varphi_k \in \Phi$ for the probability $p \subset S_{\varphi_k}$ will be referred *to as the Bayesian solution*, and the value $B^+(p, \varphi_k)$ in Bayesian solution φ_k is the *optimal Bayes value of the estimated functional.*

Determine the *Bayesian surface* of the optimal Bayes values of the estimated functional F^+ (or simply Bayesian surface) for all probabilities $p \in \Delta_n$ in the following way:

$$B^+(p) = \max_{\varphi_k \in \Phi} B^+(p, \varphi_k) \, . \tag{6.21}$$

If a decision making body has information about the Bayesian set, then they can relatively simply take the optimal decisions (using solutions after the Bayesian criteria), even if the apriori distribution of probabilities: $p = (p_1, ..., p_n)$, the states of the environment C are defined inaccurately. But the problem of constructing the Bayesian sets is a rather complicated mathematical task of partitioning $(n - 1)$-th-dimensioned simplex into subsets, especially when $n \ge 4$. Let's consider the following methods of constructing Bayesian sets of solutions.

6.3.1 Geometric method for constructing Bayesian sets

Note that this method can be used only for a small number of states of the environment, in particular, when $n = 2$ or $n = 3$ only.

In general case, the geometric method for constructing a Bayesian set in $(n – 1)$ -dimensional space of values p_1, ..., p_{n-1} for the chosen solution $\varphi_k \in \Phi$ can be described as follows:

Each of the sets S_{φ_k} will be a set of points p satisfying the following system of inequalities:

$$b^{+}_{\varphi_i\varphi_k}(\overline{p}) \geq 0, \quad i = 1,...,m, i \neq k, \ p_j \geq 0 \quad j = 1,...,n-1, \ \sum_{j=1}^{n-1} p_j \leq 1, \tag{6.22}$$

where

$$b^{+}_{\varphi_i\varphi_k}(\overline{p}) = \sum_{j=1}^{n-1} p_j (f^{+}_{ji} - f^{+}_{jk}) + (1 - \sum_{i=1}^{n-1} p_i)(f^{+}_{ni} - f^{+}_{nk}). \tag{6.23}$$

So, we have a system of $(n + m - 1)$ inequalities. From this, it follows that each of the sets S_{φ_k} is a convex closed polyhedron in $(n – 1)$-dimensional space, which is completely described by its vertices.

To find a vertex of the set S_{φ_k} it is necessary to examine various combinations, which are made of $(n – 1)$-th equality of the following form:

$$b^{+}_{\varphi_i\varphi_k}(\overline{p}) = 0, \ p_j = 0, \ \sum_{j=1}^{n-1} p_j = 1. \tag{6.24}$$

Such combinations may be equal or less than $C^{\,n-1}_{m+n-1}$. The vertex of the set S_{φ_k} will be presented by any point: $\overline{p} = (p_1,..., p_{n-1})$ satisfying the system of inequalities (6.22) and the system with $(n – 1)$-th equation (6.24) with a non zero determinant. But this method is not rational, since the number of combinations is quite wide.

Example 6.2. We consider the problem of constructing a Bayesian set $\Phi = \{\varphi_1, \varphi_2, \varphi_3\}$ for the case with three possible solutions: and two states of the environment: $\theta = \{\theta_1, \theta_2\}$, that is, when $n = 2$ and $m = 3$. Suppose, that the estimated functional is given by such a matrix:

$$\begin{array}{c|cccc} & & \varphi_1 & \varphi_2 & \varphi_3 \\ p & \theta_1 & f_{11} & f_{12} & f_{13} \\ 1-p & \theta_2 & f_{21} & f_{22} & f_{23} \end{array}.$$

Let's write the Bayes values of the functional for each of the solutions:

$$B^{+}(p,\varphi_1)=pf_{11}+(1-p)f_{21},$$

$$B^{+}(\rho,\varphi_2)=pf_{12}+(1-p)f_{22},$$

$$B^{+}(\rho,\varphi_3)=pf_{13}+(1-p)f_{23}.$$

The Bayesian surface is determined by the following formula:

$$B^{+}(\rho)=\max_{\varphi_k\in\Phi}B^{+}(\rho,\varphi_k),$$

and the boundaries of the sets are described by the points, in which the change of the surface line passes. These are the points of intersection of the corresponding Bayes values of the estimated functional, namely:

$$\Gamma_{\varphi_1\varphi_2}:\quad B^{+}(p,\varphi_1)=B^{+}(p,\varphi_2),$$

$$\Gamma_{\varphi_1\varphi_3}:\quad B^{+}(p,\varphi_1)=B^{+}(p,\varphi_3),$$

$$\Gamma_{\varphi_2\varphi_3}:\quad B^{+}(p,\varphi_2)=B^{+}(p,\varphi_3),$$

where $\Gamma_{\varphi_i\varphi_j}$ is the boundary between sets S_{φ_i} and S_{φ_j} .

Consider an example with numerical values.

Example 6.3. Assume that the estimated function is given as follows:

	φ_1	φ_2	φ_3
θ_1	10	2	5
θ_2	-3	7	4

Then the Bayes values for each decision acquire the following form:

$$B^{+}(p,\varphi_1)=10p+(-3)(1-p)=10p-3+3p=13p-3,$$

$$B^{+}(p,\varphi_2)=2p+7(1-p)=2p+7-7p=-5\rho+7,$$

$$B^{+}(p,\varphi_3)=5p+4(1-p)=5p+4-4p=p+4.$$

We construct graphs of these functions on the coordinate plane (see Figure 6.1).

A Bayesian surface $B^{+}(p)$ is marked on the drawing with thick line.

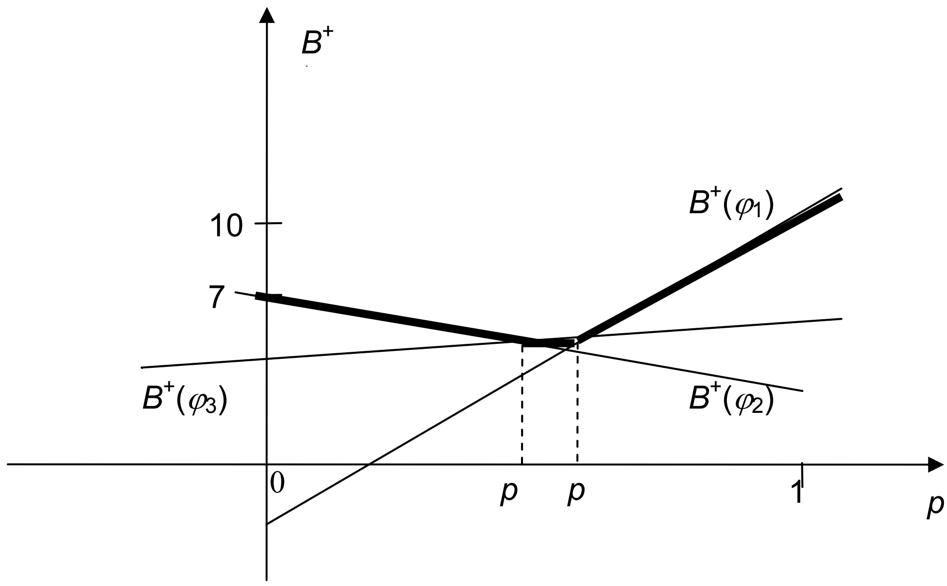


Fig. 6.1. Graphic representation of the Bayesian surface and Bayesian sets (for *Example 6.3*)

It is obvious that

$$B^+(p) = \begin{cases} B^+(p, \varphi_1), \ 0 \le p \le p_1, \\ B^+(p, \varphi_3), \ p_1 \le p \le p_2, \\ B^+(p, \varphi_2), \ p_2 \le p \le 1. \end{cases}$$

The Bayesian sets will look like this: $S_{\varphi_1} = [0, p_1]$, $S_{\varphi_2} = [p_2, 1]$, $S_{\varphi_3} = [p_1, p_2]$. To calculate the values of the probabilities p_1 and p_2, we make the following equation

$$p_1: \ B^+(p,\varphi_2) = B^+(p,\varphi_3), \qquad p_2: \ B^+(p,\varphi_3) = B^+(p,\varphi_1),$$

$$-5p + 7 = p + 4, \qquad 13p - 3 = p + 4,$$

$$-6p = -3, \qquad 12p = 7,$$

$$p_1 = \frac{1}{2}. \qquad p_2 = \frac{7}{12}.$$

Consequently, finally the Bayesian sets acquire the following form:

$$S_{\varphi_2} = \left[0; \frac{1}{2}\right], \quad S_{\varphi_3} = \left[\frac{1}{2}; \frac{7}{12}\right], \quad S_{\varphi_1} = \left[\frac{7}{12}; 1\right].$$

6.3.2 Functional method for constructing Bayesian sets

The construction of Bayesian sets, when there are several solutions, can be accomplished either by joint consideration of all possible pairs of decisions, or by successive transition from two solutions to three, from three to four etc. Therefore, there is a direct method of solving the problem of constructing Bayesian sets and multi-step method of successive increase in the number of solutions.

The direct method for constructing Bayesian sets under the condition of several possible solutions involves the sequence of actions described below.

For the initial set of solutions: $\Phi = \{\varphi_1, \ldots, \varphi_m\}$, it is necessary to make all possible pairs of solution

(φ_1, φ_2), (φ_1, φ_3), (φ_1, φ_4), …, (φ_1, φ_m) – in total $(m-1)$ pairs,

(φ_2, φ_3), (φ_2, φ_4), …, (φ_2, φ_m) – in total $(m-2)$ pairs,

(φ_1, φ_4), …, (φ_1, φ_m) – in total $(m-3)$ pairs,

(φ_1, φ_4), …, (φ_1, φ_m) – in total $(m-3)$ pairs

………………………………………………………

$(\varphi_{m-1}, \varphi_m)$ – 1 pair

and for each pair, break the simplex $P_n - 1$ into two corresponding Bayesian sets.

We denote via $S_{\varphi_k|\varphi_i}$ a Bayesian set of decisions $\varphi_k \in \Phi$ in a group of only two solutions, then

$$S_{\varphi_1} = \bigcap_{i=2}^{m} S_{\varphi_1|\varphi_i}.$$

For a part of a simplex $\mathrm{P}_{n\text{-}1} \setminus S_{\varphi_1}$ the Bayesian set S_{φ_2} can be defined as follows:

$$S_{\varphi_2} = (P_{n-1} \setminus S_{\varphi_1}) \bigcap_{i=3}^{m} S_{\varphi_2|\varphi_i}.$$

Then, we consider a part of a simplex $[P_{n-1} \setminus (S_{\varphi_1} \bigcup S_{\varphi_2})]$ for which the Bayesian set S_{φ_3} is defined with the following formula:

$$S_{\varphi_3} = [(P_{n-1} \setminus S_{\varphi_1} \cup S_{\varphi_2})] \bigcap_{i=4}^{m} S_{\varphi_3|\varphi_i}.$$

Continuing this process, then, in a similar way we can establish that:

$$S_{\varphi_k} = [(P_{n-1} \setminus \bigcup_{s=1}^{k-1} S_{\varphi_s})] \bigcap_{i=k+1}^{m} S_{\varphi_k|\varphi_i} .$$

6.3.3 Method of variation of the reference point for constructing Bayesian solutions

Assume that some probability is selected: $p^0 = (p_1^0, \dots, p_n^0)$, $p^0 \in P_{n-1}$ (reference point).

Let us describe the scheme of the method.

1. Find the Bayes values of the estimated functional in relation to the decisions $\varphi_k \in \Phi$ when $p = p^0$, namely:

$$B^+(p^0, \varphi_k) = \sum_{j=1}^{n} p_j^0 f_{jk}^+ \ , \ k = 1,2,\dots m .$$

2. Determine the solution $\varphi_{k_0} \in \Phi$ for which the following condition is fulfilled:

$$B^+(p^0) = B^+(p^0, \varphi_{k_0}) = \max_{\phi_k \in \Phi} B^+(p^0, \varphi_k) .$$

3. Calculate the discrepancies: $\delta_{k_0 k} = B^+(p^0, \varphi_{k_0}) - B^+(p^0, \varphi_k)$, for all decisions $\varphi_k \in \Phi \setminus \varphi_{k_0}$.

4. Calculate the vectors d_{kk_0} as the difference between the k-th and k_0-th columns of the estimated functional F^+ by the following rule:

$$d_{kk_0} = \begin{pmatrix} f_{1k}^+ \\ \dots \\ f_{nk}^+ \end{pmatrix} - \begin{pmatrix} f_{1k_0}^+ \\ \dots \\ f_{nk_0}^+ \end{pmatrix} .$$

5. Consider the variation: $\overline{p} = (\overline{p}_1, \dots, \overline{p}_n)$, the starting point p^0, constructed according to the rule:

$\overline{p} = p^0 + q$, where $\overline{p}_j = p_j + q_i$, $\sum_{j=1}^{n} \overline{p}_j = 1$, $\sum_{j=1}^{n} p_j^0 = 1$, $\sum_{i=1}^{n} q_i = 0.$

For each pair of solutions $(\varphi_{k_0}, \varphi_k)$, we find a scalar composition (q, d_{kk_0}), namely: $(q, d_{kk_0}) = \sum_{j=1}^{n} q_j d_{kk_0}^j$.

The boundary between the conditional Bayesian sets $S_{\varphi_{k_0}|\varphi_k}$ and $S_{\varphi_k|\varphi_{k_0}}$ is determined by calculating $(n-1)$ variational points $\overline{p}^1, ..., \overline{p}^{n-1}$ on the basis of solving $(n-1)$-th system of linear algebraic equations, with the of which vectors $q^1, ..., q^{n-1}$, can be found, that is,

$$\begin{cases} (q, d_{kk_0}) = \delta_{kk_0}, \\ q_i = 0, \qquad\qquad i = 1, 2, \ldots, n-1. \\ \sum_{l=1}^{n} q_l = 0. \end{cases}$$

6. The equation of the hyperplane $\Gamma_{\varphi_{k_0}\varphi_k}$ which is the boundary of the sets $S_{\varphi_{k_0}|\varphi_k}$ and $S_{\varphi_k|\varphi_{k_0}}$ and passes through the points $\overline{p}^1 = (\overline{p}_1^1, \ldots, \overline{p}_{n-1}^1)$ is obtained by solving the following equation:

$$\begin{vmatrix} p_1 - \overline{p}_1^1, & p_2 - \overline{p}_2^1, & \cdots & p_{n-1} - \overline{p}_{n-1}^1 \\ \overline{p}_1^2 - \overline{p}_1^1, & \overline{p}_2^2 - \overline{p}_2^1, & \cdots & \overline{p}_{n-1}^2 - \overline{p}_{n-1}^1 \\ \cdots & \cdots & \cdots & \\ \overline{p}_1^{n-1} - \overline{p}_1^1, & \overline{p}_2^{n-1} - \overline{p}_2^1, & \cdots & \overline{p}_{n-1}^{n-1} - \overline{p}_{n-1}^1 \end{vmatrix} = 0.$$

After constructing conditional Bayesian sets $S_{\varphi_{k_0}|\varphi_k}$, the Bayesian set $S_{\varphi_{k_0}}$ is defined as follows: $S_{\varphi_{k_0}} = \bigcap_{\varphi_k \in \Phi \backslash \varphi_{k_0}} S_{\varphi_{k_0}|\varphi_k}$.

Next, the process of constructing Bayesian sets is performed consistently using the described method for all solutions $\varphi_k \in \Phi$.

Example 6.4. Consider the problem of constructing Bayesian sets for the case of three possible solutions: $\Phi = \{\varphi_1, \varphi_2, \varphi_3\}$, and three states of the environment: $\Theta = \{\theta_1, \theta_2, \theta_3\}$ that is, when $n = 3$ and $m = 3$. Assume that the estimated functional is given by such a matrix:

		φ_1	φ_2	φ_3
p_1	θ_1	2	1	4
p_2	θ_2	5	2	3
$1-p_1-p_2$	θ_3	2	5	1

Write the Bayes values of the functionality for each of the possible solutions, namely:

$$B^+(p,\varphi_1) = 2p_1 + 5p_2 + 2(1-p_1-p_2) = 3p_2 + 2,$$

$$B^+(p,\varphi_2) = 1p_1 + 2p_2 + 5(1-p_1-p_2) = -4p_1 - 3p_2 + 5,$$

$$B^+(p,\varphi_3) = 4p_1 + 3p_2 + (1-p_1-p_2) = 3p_1 + 2p_2 + 1.$$

Now let's make an equation of boundaries:

1) $\Gamma_{\varphi_1,\varphi_2}$: $B^+(p,\varphi_1) = B^+(p,\varphi_2)$,

$$3p_2 + 2 = -4p_1 - 3p_2 + 5,$$
$$6p_2 = -4p_1 - 3,$$
$$p_2 = -0,67p_1 + 0,5.$$

2) $\Gamma_{\varphi_1,\varphi_3}$: $B^+(p,\varphi_1) = B^+(p,\varphi_3)$,

$$3p_2 + 2 = 3p_1 + 2p_2 + 1,$$
$$p_2 = 3p_1 - 1.$$

3) $\Gamma_{\varphi_2,\varphi_3}$: $B^+(p,\varphi_2) = B^+(p,\varphi_3)$,

$$-4p_1 - 3p_2 + 5 = 3p_1 + 2p_2 + 1,$$
$$-5p_2 = -4 + 7p_1,$$
$$p_2 = -1,4p_1 + 0,8.$$

Based on the results of the calculations, we construct the boundary image obtained on the coordinate plane (see Figure 6.2).

As you can see, the source area is divided into 6 subsets. Determine now which solution is the optimal for each of them. To do this, we choose an arbitrary point in each subset and calculate the Bayes values for each of the solutions at that point. The maximum value corresponds to a certain Bayesian set. The results of calculations are reduced to Table 6.2.

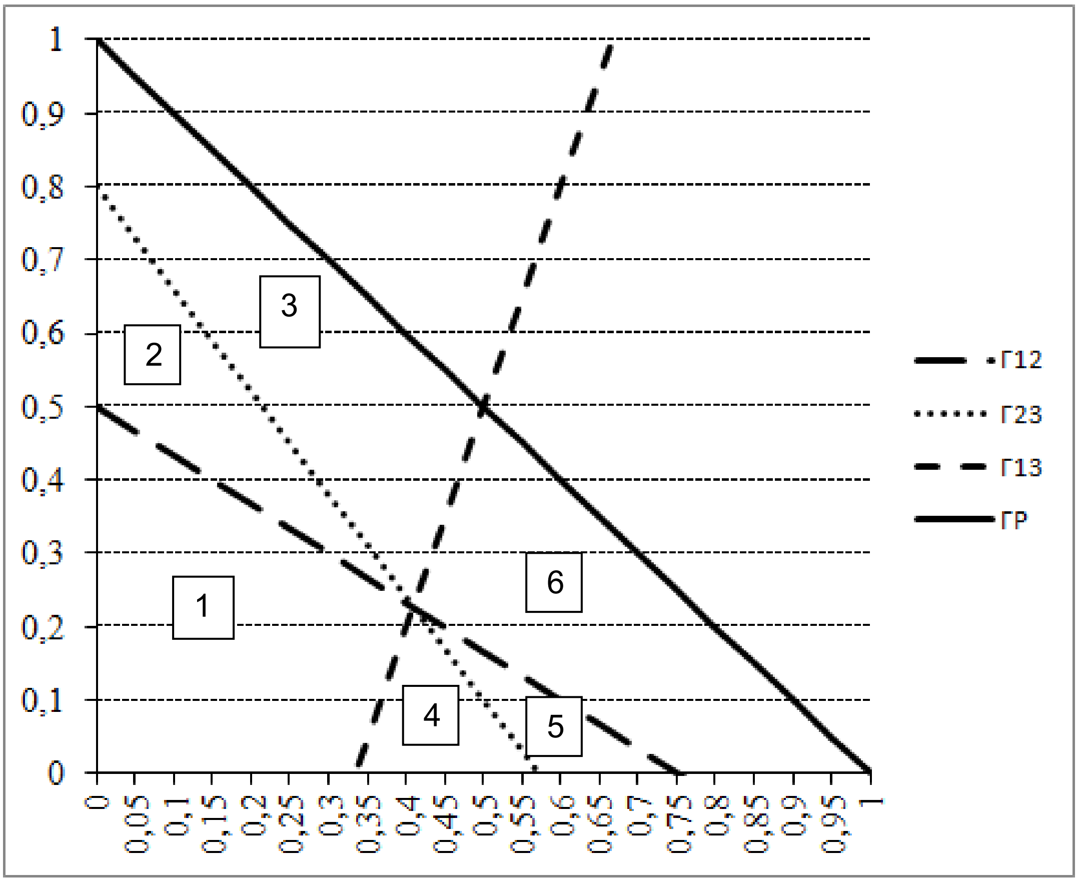


Fig. 6.2. The boundaries of subsets $S_{\varphi_{k_0}|\varphi_k}$ and $S_{\varphi_k|\varphi_{k_0}}$ obtained from the calculations in *Example 6.4*

Table 6.2

The results of calculating the Bayes values of the estimated functional in each of the subsets

Subset	Point coordinates		Bayesian values			Maximum value	Belonging to the Bayesian set
	p_1	p_2	$B^+(p,\varphi_1)$	$B^+(p,\varphi_2)$	$B^+(p,\varphi_3)$		
1	0	0	2	5	1	5	S_{φ_2}
2	0	0,6	3,8	3,2	2,2	3,8	S_{φ_1}
3	0	0,9	4,7	2,3	2,8	4,7	S_{φ_1}
4	0,45	0	2	3,2	2,4	3,2	S_{φ_2}
5	0,65	0	2	2,4	3	3	S_{φ_3}
6	0,9	0	2	1,4	3,7	3,7	S_{φ_3}

Taking into account the results obtained, we finally have the following Bayesian sets:

$$S_{\varphi_1} = \begin{cases} p_1 + p_2 \le 1, \\ p_2 \ge -0,67p_1 + 0,5, \\ p_2 \ge 3p_1 - 1, \\ p_1, p_2 \ge 0. \end{cases} \quad S_{\varphi_2} = \begin{cases} p_2 \le -0,67p_1 + 0,5, \\ p_2 \le -1,4p_1 + 0,8, \\ p_1, p_2 \ge 0. \end{cases} \quad S_{\varphi_3} = \begin{cases} p_1 + p_2 \le 1, \\ p_2 \ge -1,4p_1 + 0,8, \\ p_2 \le 3p_1 - 1, \\ p_1, p_2 \ge 0. \end{cases}$$

The graphic representation of the received Bayesian sets is shown in Figure 6.3.

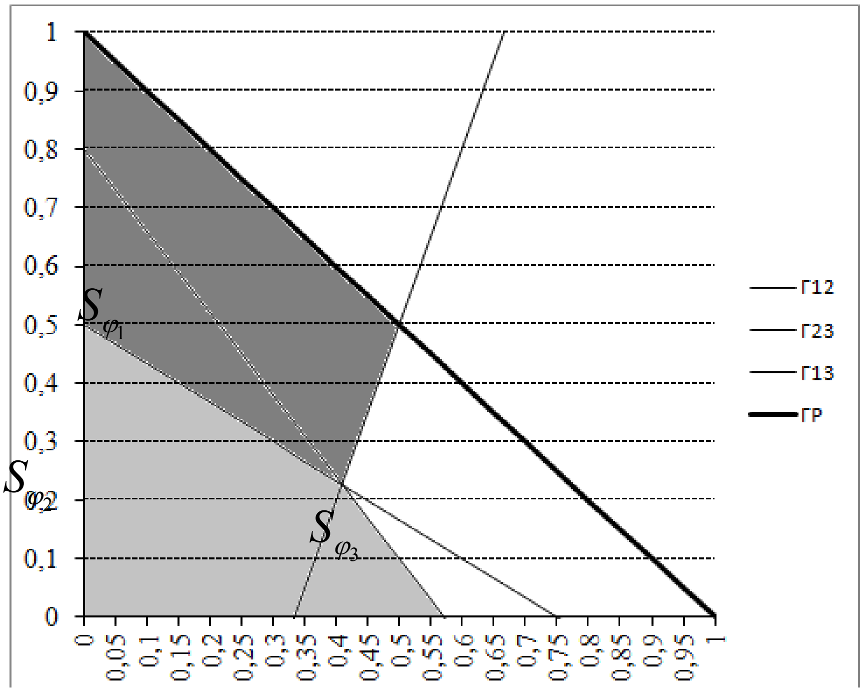


Fig. 6.3. Graphic representation of Bayesian sets (*Example* 6.4)

Example 6.5. Assume that the decision-making situation is given by a matrix:

p		φ_1	φ_2	φ_3
0,3	θ_1	2	1	4
0,5	θ_2	5	2	3
0,2	θ_3	2	5	1

The referenve point $p_0 = (0,3;\ 0,5;\ 0,2)$. We construct a Bayesian sets according to the scheme described above.

To do this, lets's follow these steps:

1. Calculate the Bayes values of the estimated functional in the control point, namely:

$$B^+(p_0, \varphi_1) = 0.3 \cdot 2 + 0.5 \cdot 5 + 0,2 \cdot 2 = 3,5,$$

$$B^{+}(p_0,\varphi_2)=0,3\cdot 1+0,5\cdot 2+0,2\cdot 5=2,3,$$

$$B^{+}(p_0,\varphi_3)=0,3\cdot 4+0,5\cdot 3+0,2\cdot 1=2,9.$$

2. Find a solution φ_{k_0} for which

$$B^{+}(p_0,\varphi_{k_0})=\max_k B^{+}(p_0,\varphi_k)=3,9.$$

In our case $k_0=1$.

3. For each solution φ_k, except φ_{k_0}, calculate the discrepancies as follows:

$$\delta_{k_0 k}=B(p_0,\varphi_{k_0})-B(p_0,\varphi_{k_0}),$$

Namely $\delta_{12}=1,2$, $\delta_{13}=0,6$, $\delta_{32}=0,6$.

4. First, we will determine the boundary between the sets S_{φ_1} and S_{φ_2}, for this purpose we calculate the following difference:

$$d_{21}=\begin{pmatrix}1\\2\\5\end{pmatrix}-\begin{pmatrix}2\\5\\2\end{pmatrix}=\begin{pmatrix}-1\\-3\\3\end{pmatrix}.$$

We introduce the variation of the control point: $\overline{p}=p_0+q$, namely:

$$\overline{p}_1=0,3+q_1,$$

$$\overline{p}_3=0,2+q_3,$$

$$\overline{p}_2=0,5+q_2.$$

Let’s calculate the scalar product (d,q), i.e.

$$(d,q)=\begin{pmatrix}-1\\-3\\3\end{pmatrix}\cdot\begin{pmatrix}q_1\\q_2\\q_3\end{pmatrix}=-q_1-3q_2+3q_3.$$

Let's write equality: $(d,q)=\delta_{12}=1,2$, and we will make a system of equations for finding the first variation, namely:

$$\begin{cases} -q_1 - 3q_2 + 3q_3 = 1,2, \\ q_1 = 0, \\ q_1 + q_2 + q_3 = 0. \end{cases}$$

Having solved it, we have got the following results: $q_1 = 0;$ $q_2 = -0,2;$ $q_3 = 0,2$.

Calculate the first point of the border, namely:

$$p_0 + q = \begin{pmatrix} 0,3 \\ 0,5 \\ 0,2 \end{pmatrix} + \begin{pmatrix} 0 \\ -0,2 \\ 0,2 \end{pmatrix} = \begin{pmatrix} 0,3 \\ 0,3 \\ 0,4 \end{pmatrix},$$

and $\overline{p}^1 = (0,3;\ 0,3;\ 0,4)$.

Let's calculate the second variation by the following system of equations:

$$\begin{cases} -q_1 - 3q_2 + 3q_3 = 1,2, \\ q_2 = 0, \\ q_1 + q_2 + q_3 = 0, \end{cases}$$

then $q_1 = -0,3;$ $q_2 = 0;$ $q_3 = 0,3$,

$$p_0 + q = \begin{pmatrix} 0,3 \\ 0,5 \\ 0,2 \end{pmatrix} + \begin{pmatrix} -0,3 \\ 0 \\ 0,3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0,5 \\ 0,5 \end{pmatrix},$$

and the second point of the border $\overline{p}^2 = (0;\ 0,5;\ 0,5)$.

The straight line passing through these two points will look like this:

$$\frac{p_1 - 0,3}{0 - 0,3} = \frac{p_2 - 0,3}{0,5 - 0,3}.$$

We convert it into a canonical form in the following way:

$$\begin{aligned} &0,2(p_1 - 0,3) = -0,3(p_2 - 0,3), \\ &0,2p_1 - 0,06 = -0,3p_2 + 0,09, \\ &0,3p_2 = -0,2p_1 + 0,15, \\ &p_2 = -0,67p_1 + 0,5. \end{aligned}$$

Consequently, the boundary $\Gamma_{\varphi_1,\varphi_2}$ between sets S_{φ_1} and S_{φ_2} is described by the following equation:

$$p_2 = -0,67\,p_1 + 0,5\,.$$

Similarly, we construct boundaries between sets S_{φ_2} and S_{φ_3}, and between S_{φ_1} and S_{φ_3}. Now we get the following results:

$$\Gamma_{\varphi_1,\varphi_3}:\ p_2 = 3p_1 - 1\,,\qquad \Gamma_{\varphi_2,\varphi_3}:\ p_2 = -1,4p_1 + 0,8.$$

As you can see, the results obtained by the two methods are the same.

6.4 Construction of decision sets relative to other criteria. Method of optimal set partition

Consider the decision-making situation $\{\Phi,\Theta,F\}$ described by such a matrix:

$$\begin{array}{cccc} & \varphi_1 & \cdots & \varphi_m \\ \theta_1 & f_{11} & \cdots & f_{1m} \\ \cdots & \cdots & \cdots & \cdots \\ \theta_n & f_{n1} & \cdots & f_{nm} \end{array}$$

and in the set of states of the environment, we know the probable distribution of probabilities and the criterion for decision-making is chosen.

You must divide the set:

$$P_{n-1} = \left\{ (p_1,\ldots,p_{n-1}) : 0 \le p_j \le 1\ ,\ j = 1,\ldots n-1,\quad \sum_{i=1}^{n-1} p_j \le 1 \right\}, \tag{6.25}$$

in its subset $S_{\varphi_k} \subset P_{n-1}$, $k = 1, \ldots m$ satisfying the following conditions:

$$S_{\varphi_i} \cap S_{\varphi_k} = \varnothing,\quad i \ne k,\quad i,k = 1,\,2,\,\ldots,\,m\,, \tag{6.26}$$

$$\bigcup_{i=1}^{m} S_{\varphi_i} = P_{n-1}\,, \tag{6.27}$$

and for $p \subset S_{\varphi_k}$, $k = 1,\ldots m$ on the optimal criterion *K* there is a solution φ_k.

In the previous section of this coursebook, the problem of constructing sets of solutions in relation to the Bayesian criterion was considered. There a geometric and functional method of constructing them were proposed. In the paper [39], the problem of constructing Bayesian sets is solved with the help of the method of optimal set partition (OSP) described in the monograph [15].

Now let's try to answer the question: is it possible to build sets of decisions in relation to other criteria? Let's consider the problem of constructing sets of solutions $S_{\varphi_k} \subset P_{n-1}$, $k = 1, \dots m$, with a relatively combined criterion of this form:

$$K(p,\varphi_k) = (1-\alpha)B^+(p,\varphi_k) - \alpha\sigma^2(p,\varphi_k), \tag{6.28}$$

where $B^+(p,\varphi_k)$ is a Bayes value for solution φ_k that corresponds to a priori distribution of probability: $p = (p_1,\dots,p_n)$; $\sigma^2(p,\varphi_k)$ – variance of values of the estimated functional, which correspond to the solution φ_k; α - parameter, $0 \le \alpha \le 1$.

Obviously, the construction of sets of solutions can be regarded as the problem of optimal partition of a simplex P_{n-1} into subsets corresponding to a possible solution. To solve it, we formulate the initial problem in the form of the problem of optimal partition of sets.

Task **A1**. To break (n – 1)-th dimensional simplex:

$$P_{n-1} = \left\{ (p_1, \dots, p_{n-1}) : 0 \le p_j \le 1,\ j = 1, \dots n-1, \quad \sum_{i=1}^{n-1} p_j \le 1 \right\}$$

in its subsets S_{φ_1}, S_{φ_2}, … S_{φ_m} in such a way that the following conditions are met:

$$S_{\varphi_i} \cap S_{\varphi_k} = \varnothing, \quad i \ne k, \quad i,k = 1, 2, \dots, m,$$

$$\bigcup_{i=1}^{m} S_{\varphi_i} = P_{n-1},$$

and functional is

$$F(S_{\varphi_1},\dots,S_{\varphi_m}) = \sum_{k=1}^{m} \int_{S_{\varphi_k}} c_k(p)\rho(p)dp, \tag{6.29}$$

reaching the maximum value.

Here $\rho(p)$ is a real, integral, integrated function;

$$c_k(p) = (1-\alpha)\sum_{i=1}^{n-1}\left((f_{ik} - f_{nk})p_i + f_{nk}\right) -$$

$$-\alpha\left(\sum_{i=1}^{n-1}\left(f_{ik}^2 - f_{nk}^2\right)p_i - \left(\sum_{i=1}^{n-1}(f_{ik} - f_{nk})p_i\right)^2 - 2f_{nk}\sum_{i=1}^{n-1}\left(f_{ik} - f_{nk}\right)p_i \right), \quad k = 1, \dots m.$$

Partition $S_{\varphi_1}, S_{\varphi_2}, \dots S_{\varphi_m}$, which is the sum of the problem A1, we will call optimal.

The formulated task A1 is a problem of OSP with fixed centers of subsets without restrictions [15].

To solve it we introduce the characteristic functions of subsets S_{φ_k}, $k=1, \dots m$, namely:

$$\lambda_k^*(p) = \begin{cases} 1, & \text{if } p \in S_{\varphi_k}, \\ 0, & \text{if } p \in S \setminus S_{\varphi_k}, \end{cases} \quad k = 1,2,\dots,m.$$

Using the terms of characteristic functions, we can write the problem A1 in this form:

Task **B1**. Find a vector function: $\lambda^*(\cdot) = \left(\lambda_1^*(\cdot),\dots,\lambda_m^*(\cdot)\right)$, which corresponds to the following condition:

$$I(\lambda^*(\cdot)) = \max_{\lambda^*(\cdot)\in\Gamma_1} I(\lambda^*(\cdot)),$$

where

$$I(\lambda(\cdot)) = \int_{P_{n-1}} \sum_{k=1}^{m} \Bigg[(1-\alpha) \sum_{i=1}^{n-1} \left((f_{ik} - f_{nk}) p_i + f_{nk} \right) -$$

$$-\alpha \left(\sum_{i=1}^{n-1} \left(f_{ik}^2 - f_{nk}^2 \right) p_i - \left(\sum_{i=1}^{n-1} (f_{ik} - f_{nk}) p_i \right)^2 - 2 f_{nk} \sum_{i=1}^{n-1} \left(f_{ik} - f_{nk} \right) p_i \right) \Bigg] \lambda_k(p) dp,$$

$$\Gamma_1 = \{\lambda(p) = (\lambda_1(p),\dots,\lambda_m(p)) \colon \sum_{k=1}^{m} \lambda_k(p) = 1 \text{ almost everywhere for } p \in P_{n-1}$$

$$\lambda_k(p) = 0 \vee 1,\ p \in P_{n-1}, k = 1,\dots,m\}.$$

Note that $F\left(S_{\varphi_1},\dots,S_{\varphi_m}\right) = I\left(\lambda(\cdot)\right)$.

Using the method of optimal partition of sets, described in the monograph [15], we obtain the solution of the problem in this form:

$$\lambda^*_k(p) = \begin{cases} 1, & \text{if } (1-\alpha)\left(B_k\right) - \alpha\left(\sigma^2{}_k\right) \geq (1-\alpha)\left(B_j\right) - \alpha\left(\sigma^2{}_j\right), \forall j = \overline{1,m} \\ 0 & \text{in other case,} \end{cases} \tag{6.30}$$

where $\lambda^*_1(p), \dots, \lambda^*_m(p)$ are characteristic functions of subsets $S^*_{\varphi_1}, \dots, S^*_{\varphi_m}$ which form the optimal partition of the simplex P_{n-1};

$$\sigma_k^2 = \sum_{i=1}^{n-1}\left(f_{ik}^2 - f_{nk}^2\right)p_i - \left(\sum_{i=1}^{n-1}(f_{ik} - f_{nk})p_i\right)^2 - 2f_{nk}\sum_{i=1}^{n-1}\left(f_{ik} - f_{nk}\right)p_i\,,$$

$$B_k = \sum_{i=1}^{n-1}(f_{ik} - f_{nk})p_i + f_{nk}\,,\ k = \overline{1,m}\,.$$

Let's assure that the optimal statement of the problem A1 will form a simplex partition in the set of solutions, that is, the following statement is true.

Statement 6.1. Subset $S_{\varphi_1}^*,...,S_{\varphi_m}^*$ which is the optimal solution of the task **A1**, form the set of solutions by the combined criterion (6.28).

Proof

Let partition $S_{\varphi_1}^*,\ldots,S_{\varphi_m}^*$ be a solution of problem A1. Consider an arbitrary subset $S_{\varphi_k}^*$, and the point $p = (p_1,\ldots,p_n) \in S_{\varphi_k}^*$. According to the necessary and sufficient condition for the optimality of the partition [15], such inequality will be fair:

$$c_k\left(p\right) \geq c_j\left(p\right), \quad j = \overline{1,m}\,. \tag{6.31}$$

Taking into account that $p_n = 1 - \sum_{k=1}^{n-1} p_k$, turn expression into function definition, namely:

$$c_k\left(p\right) = (1-\alpha)\left(\sum_{i=1}^{n-1}(f_{ik} - f_{nk})p_i + f_{nk}\right) -$$

$$-\alpha\left(\sum_{i=1}^{n-1}\left(f_{ik}^2 - f_{nk}^2\right)p_i - \left(\sum_{i=1}^{n-1}(f_{ik} - f_{nk})p_i\right)^2 - 2f_{nk}\sum_{i=1}^{n-1}\left(f_{ik} - f_{nk}\right)p_i\right) =$$

$$= (1-\alpha)\left(\sum_{i=1}^{n-1}p_i f_{ik} + (1 - \sum_{i=1}^{n-1}p_i)f_{nk}\right) -$$

$$-\alpha\left(\sum_{i=1}^{n-1}f_{ik}^2 p_i - \sum_{i=1}^{n-1}f_{nk}^2 p_i + f_{nk}^2 - f_{nk}^2 - \left(\sum_{i=1}^{n-1}\left(f_{ik} - f_{nk}\right)p_i\right)^2 - 2f_{nk}\sum_{i=1}^{n-1}\left(f_{ik} - f_{nk}\right)p_i\right) =$$

$$= (1-\alpha)\sum_{i=1}^{n}p_i f_{ik} - \alpha\left(\sum_{i=1}^{n}f_{ik}^2 p_i - f_{nk}^2 - \left(\sum_{i=1}^{n-1}\left(f_{ik} - f_{nk}\right)p_i\right)^2 - 2f_{nk}\sum_{i=1}^{n-1}\left(f_{ik} - f_{nk}\right)p_i\right) =$$

$$=(1-\alpha)\sum_{i=1}^{n} p_i f_{ik} - \alpha\left(\sum_{i=1}^{n} f_{ik}^2 p_i - \left(\sum_{i=1}^{n-1}\left(f_{ik}-f_{nk}\right)p_i + f_{nk}\right)^2\right)=$$

$$=(1-\alpha)\sum_{i=1}^{n} p_i f_{ik} - \alpha\left(\sum_{i=1}^{n} f_{ik}^2 p_i - \left(\sum_{i=1}^{n} f_{ik} p_i\right)^2\right)=(1-\alpha)B^{+}(p,\varphi_k)-\alpha\sigma^2(p,\varphi_k)$$

Then, it follows from the inequality (6.31) that

$$(1-\alpha)B^{+}(p,\varphi_k)-\alpha\sigma^2(p,\varphi_k)\geq(1-\alpha)B^{+}(p,\varphi_j)-\alpha\sigma^2(p,\varphi_j)\quad \forall j=\overline{1,m}.$$

So, $(1-\alpha)B^{+}(p,\varphi_k)-\alpha\sigma^2(p,\varphi_k)=\max\limits_{\varphi_j\in\Phi}(1-\alpha)B^{+}(p,\varphi_j)-\alpha\sigma^2(p,\varphi_j)$,

i.e. $K\left(p,\varphi_k\right)==\max\limits_{\varphi_j\in\Phi}K\left(p,\varphi_j\right)$.

Thus, the decision φ_k is optimal in relation to the combined criterion when $p=(p_1,...,p_n)$.

On the other hand, if the decision φ_k is considered optimal by the combined criterion, when $p=p_0$, then the following condition is fulfilled:

$$K(p_0,\varphi^*)=\max_{\varphi_k\in\Phi}K(p_0,\varphi_k)=\max_{\varphi_k\in\Phi}\left[(1-\alpha)\left(\sum_{i=1}^{n}p_i f_{ik}\right)-\alpha\left(\sum_{i=1}^{n}(f_{ik}-\sum_{i=1}^{n}f_{ik}p_i)^2 p_i\right)\right],$$

i.e. $(1-\alpha)\left(\sum\limits_{i=1}^{n}p_{0i}f_{ik}\right)-\alpha\left(\sum\limits_{i=1}^{n}(f_{ik}-\sum\limits_{i=1}^{n}f_{ik}p_{0i})^2 p_{0i}\right)\geq$

$$\geq(1-\alpha)\left(\sum_{k=1}^{n}p_{0i}f_{ij}\right)-\alpha\left(\sum_{i=1}^{n}(f_{ij}-\sum_{i=1}^{n}f_{ij}p_{0i})^2 p_{0i}\right),\ \forall j=\overline{1,m}.$$

Taking into account that $\sum\limits_{i=1}^{n}p_i=1$ and, converting expressions, we get the following inequality:

$$(1-\alpha)\left(\sum_{i=1}^{n-1}(f_{ik}-f_{nk})p_{0i}+f_{nk}\right)-$$

$$-\alpha\left(\sum_{i=1}^{n-1}\left(f_{ik}^2-f_{nk}^2\right)p_{0i}-\left(\sum_{i=1}^{n-1}(f_{ik}-f_{nk})p_{0i}\right)^2-2f_{nk}\sum_{i=1}^{n-1}\left(f_{ik}-f_{nk}\right)p_{0i}\right)\geq$$

$$\geq (1-\alpha)\left(\sum_{i=1}^{n-1}(f_{ij}-f_{nj})p_{0i}+f_{nj}\right)-$$

$$-\alpha\left(\sum_{i=1}^{n-1}\left(f_{ij}^2-f_{nj}^2\right)p_{0i}-\left(\sum_{i=1}^{n-1}(f_{ij}-f_{nj})p_{0i}\right)^2-2f_{nj}\sum_{i=1}^{n-1}\left(f_{ij}-f_{nj}\right)p_{0i}\right), \quad \forall j=\overline{1,m}.$$

And according to formula (6.30), this means that $p_0 \in S_{\varphi_k}$.

So, subsets $S^*_{\varphi_1},...,S^*_{\varphi_m}$ form the set of decisions in relation to the combined criterion. The statement is proven.

We formulate the algorithm for solving the task **A1**, which is based on the OSP method [40].

Algorithm A1

1. We put the set P_{n-1} in a parallelepiped P which sides are parallel to the axes of the Cartesian coordinate system.
2. We cover the parallelepiped P with the rectangular grid.
3. We set the function in the nodes of the grid by the following rule:

$$\rho(p)=\begin{cases}1, & \text{if } p\in P_{n-1},\\ 0, & \text{if } p\notin P_{n-1}.\end{cases}$$

4. We calculate the value of the characteristic functions $\lambda_k^*(p), k=\overline{1,m}$ in the nodes of the grid with the formula (6.30).
5. When $\lambda_k^*(p)=1$, the point p belongs to the set S_{φ_k}. Otherwise, no.
6. To check the correctness of calculations, we calculate the value of the target function $F(S_{\varphi_1},...,S_{\varphi_m})$ in the node p.

Let's illustrate the example of the application of the described algorithm in the example that follows.

Example 6.6. Let's consider the decision-making situation, which is described by such a matrix:

	φ_1	φ_2	φ_3	φ_4
θ_1	4,05	3,29	2,93	0,99
θ_2	3,79	4,12	4,05	4,19
θ_3	3,23	3,54	4,75	5,95

It is necessary to construct sets of solutions corresponding to the combined criterion (6.28).

The result of the algorithm A1 is shown in Figure 6.4. When $\alpha = 0$, we obtain the Bayes sets (Figure 6.4, *a*), if $\alpha = 1$ – the sets corresponding to the dispersion minimum criterion (Figure 6.4, *b*), the results for the combined criterion are $\alpha = 0,3$ (Figure 6.4, *c*) and $\alpha = 0,7$ (Figure 6.4, *d*).

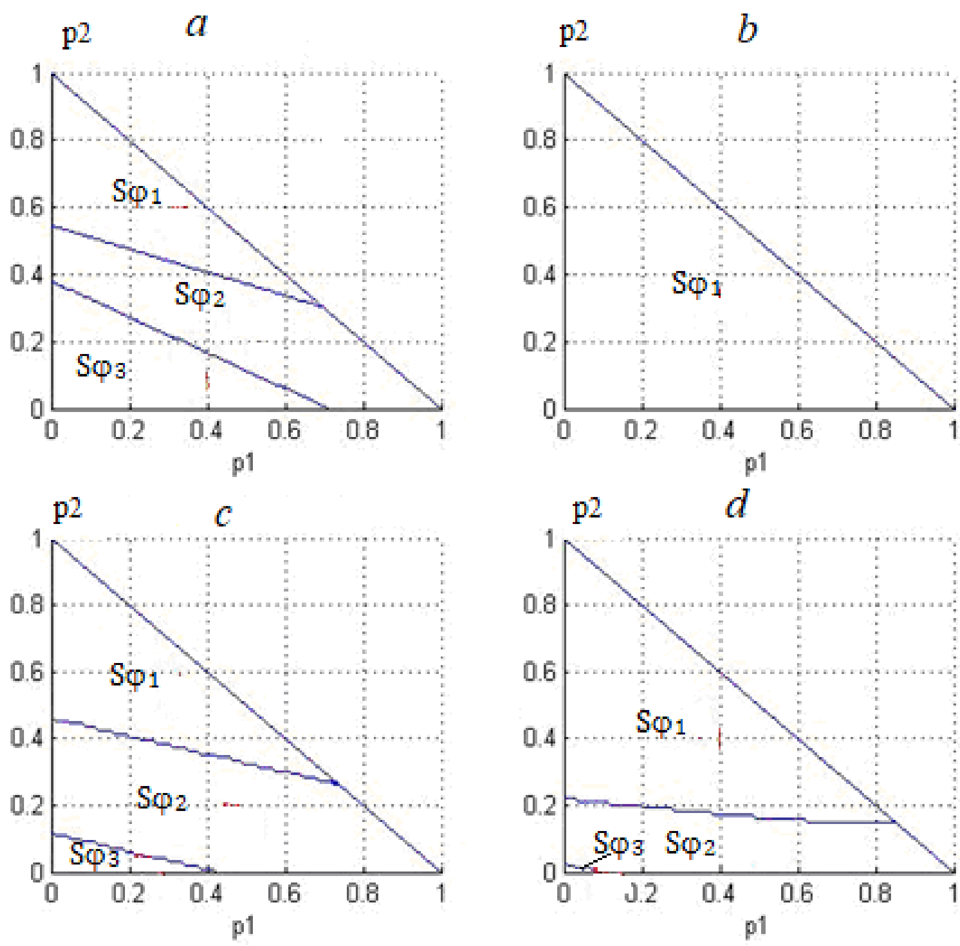


Fig. 6.4. Graphical representation of sets of decisions by the combined criterion, obtained using Algorithm A1

It is obvious that if the criterion of the minimum dispersion ($\alpha = 1$) is applied, then, under an arbitrary a priori probability distribution, the solution φ_1 will be optimal, at the same time φ_4, the solution will not be used in any case and can be excluded from consideration. The use of other solutions depends on the a priori distribution of probability and the type of criterion applied.

Note that the described algorithm can also be extended to situations with more than three states of the environment, and the construction of sets of solutions allows us to find the optimal solution even in the case of an uncertain a priori probability distribution or at least to evaluate a possible error.

6.5 Criteria for making decisions in conditions of complete uncertainty

Let's now consider the decision criteria that are used in other information situations. Let's start with the situation of the I_4, characteristic of the unknown probability distribution: $p = (p_1, p_2 \dots p_n)$, $p_j = p(\theta = \theta_j)$, $\sum_{j=1}^{n} p_j = 1$, in the set of states of the environment $\theta_1 \dots \theta_n$ and the absence of active counteraction of the environment to decision-making purposes. In the specific sense, this situation corresponds to the model of passive "behaviour" of the environment in the theory of statistical decisions. In other words, it reflects the complete absence of data about the environment in the management body. In real conditions, such situations are associated with the introduction of new equipment or the implementation of new samples of goods, when the demand for products is completely unknown etc.

Let's consider the decision criteria which can be applied to this situation. Conditionally, they can be divided into two groups: criteria of integral values and evaluation criteria.

Let's characterize varieties of integral criteria.

Criteria of the maximum measure of Bayes sets. We have a decision-making situation $\{\Phi, \Theta, F\}$. Let's denote through $S_{\varphi_1}, S_{\varphi_2}, \dots S_{\varphi_m}$ the Bayesian sets of solutions $\varphi_1, \varphi_2, \dots \varphi_m$, respectively, and through $\mu(S_{\varphi_1}), \mu(S_{\varphi_2}), \dots \mu(S_{\varphi_m})$ the measure of these sets. As the probability distribution is unknown in situation *I4,* then the principle of the maximum measure of the Bayesian sets can be considered appropriate for choosing a solution. It corresponds to the assumption that the a priori distribution of the Bayesian set which has a greater measure is more likely to be true for the environment *C.*

Thus, the optimal is considered the solution $\varphi_{k_0} \in \Phi$ which satisfies the following condition:

$$\mu(\varphi_{k_0}) = \max_{\varphi_k \in \Phi} \mu(\varphi_k). \tag{6.32}$$

The disadvantage of this criterion is that the optimal solution φ_{k_0} may not always satisfy the desired condition for the decision-making body, i.e.

$$\int_{S_{\varphi_0}} B^+(\varphi_0, p)\,dp \ge \int_{S_{\varphi_k}} B^+(\varphi_k, p)\,dp,$$

where is the value $\int\limits_{S_{\varphi_k}} B^+(\varphi_k, p)\,dp$ characterizes the integral Bayesian value of the estimated functional.

The criterion for the maximum integral Bayesian value of the estimated functional. The *integral Bayes value of the estimated functional* in the solution φ_k is referred to the value $\int\limits_{S_{\varphi_k}} B^+(\varphi_k, p)\,dp$.

In accordance with the above criterion, the solution φ_{k_0} that satisfies the following condition will be optimal:

$$\int\limits_{S_{\varphi_0}} B^+(\varphi_0, p)\,dp = \max_{\varphi_k \in \Phi} \int\limits_{S_{\varphi_k}} B^+(\varphi_k, p)\,dp. \qquad (6.33)$$

The disadvantage of this criterion is that the optimal solution φ_{k_0} may not always be in line with the desired decision-making body condition, namely:

$$\mu\left(\varphi_{k_0}\right) \geq \mu\left(\varphi_k\right).$$

Criterion of maximum integral potential. The disadvantages of the criteria described above may be somewhat neutralized by applying the principle of choice based on the concept of the potential solution.

The *integral potential* of the solution $\varphi_k \in \Phi$ will be referred to the following value:

$$\pi_{\varphi_k} = \frac{\int\limits_{S_{\varphi_k}} B^+(\varphi_k, p)\,dp}{1 - \mu(\varphi_k)/\mu(P_{n-1})}. \qquad (6.34)$$

The optimal criterion for maximum potential is considered to be a solution $\varphi_{k_0} \in \Phi$ that satisfies the following condition:

$$\pi_{\varphi_{k0}} = \max_{\varphi_k \in \Phi} \pi_{\varphi_k}.$$

The described criterion can be considered a convolution of the previous two criteria.

Consider the example of the criteria described above.

Example 6.7. Let the decision-making situation be described by such a matrix

$$\begin{array}{ccc|c} & \varphi_1 & \varphi_2 & \varphi_3 \\ \theta_1 & 10 & 2 & 5 \\ \theta_2 & -3 & 7 & 4 \end{array}$$

The Bayesian sets corresponding to this situation was constructed in 6.3.1 (see Example 6.3), namely: $S_{\varphi_1} = \left[\frac{7}{12}; 1\right]$, $S_{\varphi_2} = \left[0; \frac{1}{2}\right]$, $S_{\varphi_3} = \left[\frac{1}{2}; \frac{7}{12}\right]$. Determine the measure of these sets. It's easy to make sure that:

$$\mu\left(S_{\varphi_1}\right) = \frac{5}{12};\ \mu\left(S_{\varphi_2}\right) = \frac{1}{2};\ \mu\left(S_{\varphi_3}\right) = \frac{1}{12}.$$

As we see, the criterion for the maximum measure of Bayesian sets φ_2 will be the optimal solution since the set of greatest measure corresponds it.

Now calculate the integral Bayesian values for each of the solutions. As shown above (see Example 6.3), the Bayesian values of the estimated functionality are as follows:

$$B^+(\varphi_1, p) = 13p - 3,\ B^+(\varphi_2, p) = -5p + 7,\ B^+(\varphi_3, p) = p + 4,$$

$$\int\limits_{S_{\varphi 1}} B^+(\varphi_1, p)\,dp = \int\limits_{\frac{7}{12}}^{1} (13p - 3)\,dp = 13p^2 - 3p\Big|_{\frac{7}{12}}^{1} = 7{,}326,$$

$$\int\limits_{S_{\varphi 2}} B^+(\varphi_2, p)\,dp = \int\limits_{0}^{0{,}5} (-5p + 7)\,dp = -5p^2 + 7p\Big|_{0}^{0{,}5} = 2{,}25,$$

$$\int\limits_{S_{\varphi 3}} B^+(\varphi_3, p)\,dp = \int\limits_{0{,}5}^{\frac{7}{12}} (p + 4)\,dp = p^2 + 4p\Big|_{0{,}5}^{\frac{7}{12}} = 0{,}4236.$$

As the results of calculations show, the solution optimized by the criterion of maximizing the integral Bayesian value will be φ_1.

Now compute the value of the integral potential for each of the solutions by the formula (6.34), namely: $\pi_{\varphi_1} = 17{,}5824$, $\pi_{\varphi_2} = 4{,}5$, $\pi_{\varphi_3} = 0{,}4621$. With regard to this criterion, the solution φ_1 will also be optimal.

Bernoulli – Laplace Criterion. The application of the above criterion in conditions of complete uncertainty is based on the principle of inadequate reason, its essence that when there is no reason to consider any state of the environment more likely than others, then a priori probabilities should be considered equal, that is,

$\hat{p} = (\hat{p}_1, \hat{p}_2 ... \hat{p}_n)$, $\hat{p}_j = \dfrac{1}{n}$, $j = 1, 2, \ldots n$, and after they have been determined, the decision can be taken according to the criteria of the information situation I_1.

The Bernoulli-Laplace criterion implies the use of the Laplace principle of insufficient base and Bayesian criterion, in particular, the optimal criterion for this criterion will be a solution φ_{k_0} that satisfies the following condition:

$$B^+(\hat{p}, \varphi_{k_0}) = \max_{\varphi_k \in \Phi} B^+(\hat{p}, \varphi_k) = \max_{\varphi_k \in \Phi} \frac{1}{n} \sum_{j=1}^{n} f_{jk} .$$

We will analyze the solution obtained on the basis of the matrix of the estimated functional. Obviously, a solution φ_k is better than a solution when φ_i such a difference is an integral part: $B^+(\hat{p}, \varphi_k) - B^+(\hat{p}, \varphi_i) = \dfrac{1}{n} \sum_{j=1}^{n} (f_{jk}^+ - f_{ji}^+)$. Then you can determine the necessary and sufficient condition that the solution φ_k will be optimal, namely:

$$\min_{\substack{\varphi_i \in \Phi \\ i \neq k}} \left\{ \frac{1}{n} \sum_{j=1}^{n} (f_{jk}^+ - f_{ji}^+) \right\} \geq 0 .$$

More information on the study of this criterion can be found in literary sources [35].

Let’s apply this criterion in an example.

Example 6.7. Let the decision-making situation be described by such a matrix:

	φ_1	φ_2	φ_3
θ_1	1	4	2
θ_2	5	3	1
θ_3	2	3	5

It is necessary to find the optimal solution according to Bernoulli-Laplace criterion.

Solving

First, we define the a priori distribution of probability. Since the conditions of the problem predict three states of the environment, then $p_1 = p_2 = p_3 = \dfrac{1}{3}$. Now calculate the Bayesian values for each of the solutions, namely:

$$B^+(\varphi_k) = \frac{1}{3} \sum_{i=1}^{3} f_{ik}^+ .$$

In numerical terms $B^{+}(\varphi_1)=\frac{7}{3}$, $B^{+}(\varphi_2)=\frac{10}{3}$, $B^{+}(\varphi_3)=\frac{8}{3}$.

As we see, the solution φ_2 is optimal.

6.6 Criteria for making decisions in conditions of antagonistic environment behaviour

Let's now consider the decision criteria in conditions of antagonistic behavior of the environment (informational situation I_5). In other words, the environment actively opposes the decision-making objectives, that is, from all its states, it chooses exactly those in which the evaluative functional acquires its worst values. That is why, in this situation, the choice of solution, which allows you to get the guaranteed values of the estimated functionality, will be rational. This can be achieved using Wald and Sevig's criteria.

Wald criterion[1] (maximin principle) is used when the estimated functionality describes the efficiency, benefits, i.e. it has a positive ingredient: $F = F^{+}$. At the same time, the choice φ_{k_0} of a solution satisfying such a condition is considered rational:

$$f_{k_0} = \max_{\varphi_k \in \Phi} \min_{\theta_j \subset \theta} f_{jk}^{+}.$$

In other words, the chosen solution provides the maximum benefit in the worst situation.

Example 6.8. The situation of decision-making is given by such a matrix:

	φ_1	φ_2	φ_3	φ_4
θ_1	2	1	7	8
θ_2	8	6	3	3
θ_3	2	10	4	0
θ_4	2	3	11	4
θ_5	16	4	7	10

It is necessary to find a rational solution, taking into account the antagonistic behavior of the environment and the necessity of obtaining the maximal value by the estimated functionality.

Solving

Apply the Wald criterion. To do this, we define the smallest element in each column, and then we will select the largest among them.

[1] This criterion was developed by Hungarian mathematician Abraham Wald in the early 1940s

$$f_1 = \min_{\theta_j \subset \theta} f_{j1}^+ = \min\{2,8,2,2,16\} = 2,$$

$$f_2 = \min_{\theta_j \subset \theta} f_{j2}^+ = \min\{1,6,10,3,4\} = 1,$$

$$f_3 = \min_{\theta_j \subset \theta} f_{j3}^+ = \min\{7,3,4,11,7\} = 3,$$

$$f_4 = \min_{\theta_j \subset \theta} f_{j4}^+ = \min\{8,3,0,4,10\} = 0,$$

$$\max_{\varphi_k \in \Phi} \min_{\theta_j \subset \theta} f_{jk}^+ = \max\{2,1,3,0\} = 3.$$

The maximum value of the functional corresponds to the solution φ_3, so its choice can be considered optimal.

The advantages of the Wald criterion include the fact that it is "extremely conservative in situations where conservatism may take place" [35], but its disadvantage is that it proceeds from the assumption that an adversary is a perfect master who always finds the best (for self) solution, and this does not always correspond to reality.

However, objections may be raised against this criterion. Let’s illustrate it on an example.

Example 6.9. Let's take the decision- making situation which is described by the following matrix.

$$F^+ = \begin{pmatrix} & \varphi_1 & \varphi_2 \\ \theta_1 & 0 & 1 \\ \theta_2 & 100 & 1 \end{pmatrix}.$$

It's easy to make sure that Wald's criterion is the best solution φ_2 however, the average value of the functionality for the solution φ_1: $B^+(\varphi_1) = 100(1-p)$, will be greater than the average decision value φ_2: $B^+(\varphi_2) = 1-p$, for all a priori probability distributions: $p_1 = p$, $p_2 = 1-p$, when $0 \le p \le 1-10^{-2}$. But despite this, the choice of solution will be justified if the environment acts as a conscious opponent to the management body.

All the given above suggests that, under certain conditions, it may be advisable to introduce additional constraints which, for example, are based on the Bernoulli-Laplace criterion, that is,

$$f_{k_0} = \max_{\varphi_k \in \Phi} \min_{\theta_j \subset \theta} f_{jk}^+,$$

$$B^+(\frac{1}{n}, \varphi_k) \ge B_0.$$

The ***Savage criterion*** (minimax risk) was proposed by Leonard Savage in 1951. It is one of the main ones in the frequency of its use in the theory of statistical decisions. It is used when the estimated functional shows the loss or risk, that is, $F = F^{-}$. In this case, the optimal solution will be the following:

$$f_{k_0} = \min_{\varphi_k \in \Phi} \max_{\theta_j \subset \theta} f_{jk}^{+}.$$

By using this criterion, as in the Wald criterion, it would be advisable to restrict the value of the estimated functional to the Bayesian value, namely:

$$B^{-}(\frac{1}{n}, \varphi_k) \le B_0.$$

Note that the Savage criterion allows to "soften" the conservatism of the minimax criterion by replacing the winning matrix with the loss matrix, which is defined as follows:

$$a^{-}\left(\theta_j, \varphi_k\right) = \max_{\varphi_k}\left\{a^{+}\left(\theta_j, \varphi_k\right)\right\} - a^{+}\left(\theta_j, \varphi_k\right),\ \theta_j \in \Theta,\ \varphi_k \in \Phi.$$

Take, for example, the prize matrix from Example 6.9:

$$F^{+} = \begin{pmatrix} & \varphi_1 & \varphi_2 \\ \theta_1 & 0 & 1 \\ \theta_2 & 100 & 1 \end{pmatrix}.$$

Optimum for a minimax criterion will be a solution with a guaranteed win per unit. Let's see what result will be when replacing this matrix with a loss matrix.

According to the above transformations, the matrix becomes of the following form:

$$F^{-} = \begin{pmatrix} & \varphi_1 & \varphi_2 \\ \theta_1 & 1 & 0 \\ \theta_1 & 0 & 99 \end{pmatrix},$$

and, in accordance with the Savage criterion, the choice of solution φ_1 will be rational.

The main objection to this criterion are as following: when the decision $\varphi_{k_0} \in \Phi$ is optimal for the Savage criterion, and we will remove the optimal solution Φ from a plurality of solutions $\varphi_k \neq \varphi_{k_0}$ then in a new set $\Phi \backslash \varphi_k$ the decision φ_{k_0} may not be optimal.

6.7 Criteria for making decisions in conditions of partial uncertainty

Informational situation I_6 is characterized by the presence of factors that determine two types of environmental behaviour.

The first is characterized by the management body having some information about the true distribution of probabilities on a plurality of environmental conditions. Although it is not enough to determine accurately the informational situation, it is possible to establish a degree of optimism-pessimism about the behaviour of the environment.

The second type assumes that the management body has information about the state of the environment, which is an intermediate between the information situations I_1 and I_5, in other words, there is complete or partial knowledge about the distribution of probabilities on the set of states of the environment and its antagonistic behaviour.

Consider criteria that may be useful in such situations.

Hurwitz Criterion. It is built on the basis of the desire of the management body to take into account not only the worst situation for them (as the Wald and Savage criteria), but also the best one. That is why it is a weighted combination of maximax and maximin criteria.

The essence of the Hurwitz criterion is to find an optimal solution that satisfies the following condition:

$$\lambda \min_{\theta_j \in \Theta} f^{+}_{jk_0} + (1-\lambda) \max_{\theta_j \in \Theta} f^{+}_{jk_0} = \max_{\varphi_k \in \Phi} \{\lambda \min_{\theta_j \in \Theta} f_{ik} + (1-\lambda) \max_{\theta_j \in \Theta} f_{jk}\}, 0 \le \lambda \le 1.$$

When $\lambda = 1$, then the Hurwitz criterion coincides with the Wald criterion, and when $\lambda = 0$ – with a maximaximum criterion, which meets the conditions of the most favourable state of the environment. The real state of the environment is somewhere between these extreme cases which is characterized by magnitude $\lambda \in [0;1]$. Together with the described criterion, it is also possible to apply the modified Hurwitz criterion, when each solution $\varphi_k \in \Phi$ corresponds to its value of the coefficient $\lambda_k \in [0;1]$. In particular, the solution that satisfies the following condition is considered optimal:

$$\lambda_{k_0} \min_{\theta_j \in \Theta} f^{+}_{jk_0} + (1-\lambda_{k_0}) \max_{\theta_j \in \Theta} f^{+}_{jk_0} = \max_{\varphi_k \in \Phi} \{\lambda_k \min_{\theta_j \in \Theta} f_{ik} + (1-\lambda_k) \max_{\theta_j \in \Theta} f_{jk}\},$$

$$0 \le \lambda_k \le 1, \quad k = 1, 2, \ldots m.$$

Let's dwell on the practical application of this criterion.

Example 6.10. Let the decision-making situation be described by such a matrix:

F^+	φ_1	φ_2	φ_3	φ_4	φ_5
θ_2	1	3	0	2	7
θ_3	3	1	10	8	1
θ_4	5	5	4	5	6
θ_5	3	7	3	6	2
θ_6	8	2	5	4	8

.

At the same time, the level of OSS optimism-pessimism is $\lambda = 0,7$. We will choose the optimal solution by the Hurwitz criterion. To do this, we first compute the value of Hurwitz's indicator for each of the solutions with the following formula:

$$f_{\lambda k} = \lambda \min_{\theta_j \in \Theta} f_{ik} + (1-\lambda) \max_{\theta_j \in \Theta} f_{jk} .$$

For convenience, write the results of the calculation in the form of the following table:

F^+	φ_1	φ_2	φ_3	φ_4	φ_5
θ_2	1	3	0	2	7
θ_3	3	1	10	8	1
θ_4	5	5	4	5	6
θ_5	3	7	3	6	2
θ_6	8	2	5	4	8
$\min_{\theta_j \in \Theta} f_{jk}$	1	1	0	2	1
$\max_{\theta_j \in \Theta} f_{jk}$	8	7	10	8	8
$f_{\lambda k}$	3,1	2,8	3	3,8	3,1

As we see, the maximum value of the Hurwitz index corresponds to the solution φ_4, so choosing it in these conditions can be considered optimal.

Consider the example of a possible objection to the Hurwitz criterion. Assume that the decision-making situation is described by such a matrix:

$$F^{+}=\begin{pmatrix} & \varphi_1 & \varphi_2 \\ \theta_1 & 0 & 1 \\ \theta_2 & 1 & 0 \\ \theta_3 & 1 & 0 \\ \vdots & \vdots & \vdots \\ \theta_{100} & 1 & 0 \end{pmatrix}.$$

By the Hurwitz criterion, both solutions have the same values. Therefore, they are optimal, but from the perspective of the given matrix of the estimated functional, the solution φ_1 is much better than φ_2. This fact can be taken into account if having introduced for each decision, $\varphi_k \in \Phi$, being investigated for optimality by the Hurwitz criterion, the restriction of this kind:

$$B^{+}\left(\frac{1}{n},\varphi_k\right) \geq B_0^{+},$$

where B_0^{+} is a given value.

Let's consider now the question of choosing the value of the coefficient $\lambda \in [0;1]$. Obviously, it corresponds to a certain degree of optimism-pessimism of DMP. The greater the confidence of the DMP in respect to one of the extreme cases of environmental behaviour, the closer to 0 or 1 will be the value λ. Value $\lambda = \frac{1}{2}$, being the equilibrium point of the gap, indicates that the DMP considers at equal degrees the environment as antagonistic and as that it will maximally contribute to the decision-making objectives. In the general case, the optimal solution on the Hurwitz criterion is a function of λ. Let's illustrate this fact in an example.

Example 6.11. Assume that the decision-making situation is described by such a matrix:

$$\begin{array}{ccccc} & \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 \\ \Theta_1 & 1 & 5 & 3 & 0 \\ \Theta_2 & 2 & 2 & 2 & 4 \\ \Theta_3 & 4 & 2 & 3 & 6 \\ \Theta_4 & 0 & 4 & 3 & 1 \end{array}.$$

Calculate the value of Hurwitz's criterion $f_{\lambda k}$ for each decision $\varphi_k \in \Phi$ with the following formula:

$$f_{\lambda k} = \lambda \min_{\theta_j \in \Theta} f_{jk} + (1-\lambda) \max_{\theta_j \in \Theta} f_{jk}.$$

Also

$$f_{\lambda 1} = \lambda \min_{\theta_j \in \Theta} f_{j1} + (1-\lambda) \max_{\theta_j \in \Theta} f_{j1} = \lambda \cdot 0 + 4(1-\lambda) = 4 - 4\lambda;$$

$$f_{\lambda 2} = \lambda \min_{\theta_j \in \Theta} f_{j2} + (1-\lambda) \max_{\theta_j \in \Theta} f_{j2} = 2\lambda + 5(1-\lambda) = 5 - 3\lambda;$$

$$f_{\lambda 3} = \lambda \min_{\theta_j \in \Theta} f_{j3} + (1-\lambda) \max_{\theta_j \in \Theta} f_{j3} = 2\lambda + 3(1-\lambda) = 3 - \lambda;$$

$$f_{\lambda 4} = \lambda \min_{\theta_j \in \Theta} f_{j4} + (1-\lambda) \max_{\theta_j \in \Theta} f_{j4} = 0 \cdot \lambda + 6(1-\lambda) = 6 - 6\lambda.$$

Construct the graphs of the obtained dependencies (see Figure 6.4).

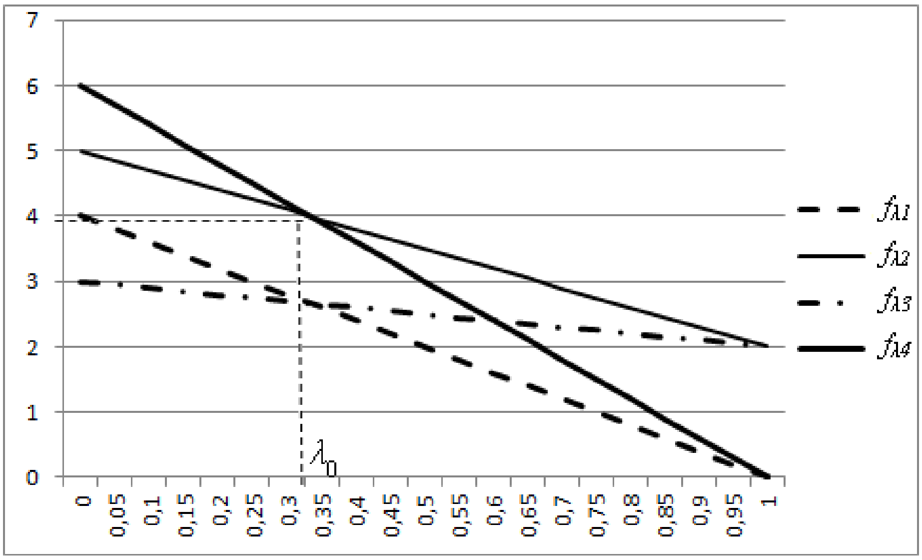


Fig. 6.4. Curves and sets of Hurwitz for Example 6.10

It is obviously that from the perspective of different values of the indicator λ, the solutions φ_2 and φ_4 will be optimal, but φ_1 and φ_3 are non-optimal, no matter what values they have. Thus, the set is divided into two subsets: Δ_{φ_2} and Δ_{φ_4}, and $\Delta_{\varphi_2} = [0; \lambda_0]$, $\Delta_{\varphi_4} = [\lambda_0; 1]$.

Let's find these sets. To do this, we will calculate the value of the parameter λ_0 as the point of intersection of straight lines $f_{\lambda 2}$ i $f_{\lambda 4}$, also: $\lambda_0 = \frac{1}{3}$. So, $\Delta_{\varphi_2} = \left[0; \frac{1}{3}\right]$, $\Delta_{\varphi_4} = \left[\frac{1}{3}; 1\right]$. When $\lambda \in \Delta_{\varphi_2} = \left[0; \frac{1}{3}\right]$, then the optimal Hurwitz criterion will be the solution φ_2, and when $\lambda \in \Delta_{\varphi_4} = \left[\frac{1}{3}; 1\right]$, the solution φ_4 will be optimal.

Consequently, the following definition can be formulated. A *set of Hurwitz* is referred to a set Δ_{φ_k} , which satisfies the following condition:

$$\Delta_{\varphi_k} = \left\{ \lambda \in [0;1] \middle| f_{\lambda k} = \max_{\varphi_i \in \Phi} f_{\lambda i} \right\},\ \varphi_k \in \Phi,$$

Moreover,

$$\bigcup_{\varphi_k \in \Phi} \Delta_{\varphi_k} = [0;1],\quad \Delta_{\varphi_k} \cap \Delta_{\varphi_i} = \varnothing, \text{ when } \varphi_i \neq \varphi_k.$$

Hurwitz curve we will call a jagged line $\Gamma^+(\lambda)$, defined as follows:

$$\Gamma^+(\lambda) = \left\{ f_{\lambda k}, \text{коли } \lambda \in \Delta_{\varphi_k}, k = 1, 2, \ldots m \right\}.$$

It may be obvious that the curve and the sets of Hurwitz are similar to Bayesian curves and sets. Therefore, based on them, we can formulate the criteria for the maximum measure of sets of Hurwitz, the maximal integral value of the Hurwitz index, and also the maximum of the integral potential.

Hodges - Lehman Criterion. This criterion takes into account the assumption that, in actual decision-making problems and tasks, actual information about the situation is often between complete ignorance and the availability of accurate data regarding the a priori probability distribution. For example, the a priori distribution may seem fairly reliable, but it is still insufficiently reliable to base the decisions on it.

The use of the Hodges-Lehman criterion allows you to take into account the information that DMP has and at the same time provides some level of guarantee in the event if it is not accurate. In a certain sense, this criterion is a "mix" of Bayes and Wald's criteria.

Let's consider the situation of decision-making $\{\Phi, \Theta, F\}$ when the estimated functional is given in the form of risks. Let's call a decision φ_{k_0} *a limited Bayesian solution* in respect to the given priori distribution $p \in \Delta_n$, if $B^-\left(\varphi_{k_0}, p\right) = \min_{\varphi_k \in \Phi} B^-\left(\varphi_k, p\right)$ and besides, there is such an inequality: $f^-_{jk_0} \leq f_0$, where f_0 is a given threshold value of the functional.

A limited Bayesian solution can also be defined by the following condition:

$$\min_{\varphi_k \in \Phi} \left\{ \lambda B^-\left(\varphi_k, p\right) + (1 - \lambda) \max_{\theta_j \in \Theta} f^-_{jk} \right\},$$

where constant $\lambda \in [0;1]$ and reflects the degree of confidence in the information that DMP has.

Choosing the optimal solution for the Hodges-Lehman criterion is convenien, using the following algorithm:

1. Determine the minimax risk, i.e. $f = \min_{\varphi_k \in \Phi} \max_{\theta_j \in \Theta} f_{jk}^{-}$.

2. Taking into account the calculated value of risk and the conditions for making a decision, choose the size of the maximum permissible risk f_0, and $f_0 \geq f$;

3. Choose the solution φ_{k_0}, which is the best to the Bayesian criterion for the admissible value of the a priori distribution $p_0 \in \Delta_n$, when the following condition is fulfilled: $f_0 \geq \max_{\theta_j \in \Theta} f_{jk}$.

Now apply the criterion considered in an example.

Example 6.11. Let the decision-making situation be given by such a matrix:

F^+	φ_1	φ_2	φ_3	φ_4	φ_5
θ_2	4	3	0	2	7
θ_3	3	5	10	8	3
θ_4	5	5	4	5	6
θ_5	3	7	3	6	4
θ_6	8	6	5	4	8

The priori probability distribution is estimated as follows:

$$p_0 = (0{,}1;\, 0{,}3;\, 0{,}2;\, 0{,}1;\, 0{,}3).$$

Determine which solution will be optimal for the Hodges-Lehman criterion.

To this end, we write the matrix of the estimated functional in the form of risks or losses, for which we first find the maximum value of the estimated functional, that is $f_{\max} = \max_{\varphi_k \in \Phi} \max_{\theta_j \in \Theta} f_{jk}^{+} = 10$. Now the loss can be estimated as follows: $f_{jk}^{-} = f_{\max} - f_{jk}^{+}$.

As a result, the matrix of the estimated functional acquires the following form:

F^-	φ_1	φ_2	φ_3	φ_4	φ_5
θ_2	6	7	10	8	3
θ_3	7	5	0	2	7
θ_4	5	5	6	5	4
θ_5	7	3	7	4	6
θ_6	2	4	5	6	2

Next, we will perform calculations according to the above algorithm.

1. Calculate the minimax risk: $f = \min\limits_{\varphi_k \in \Phi} \max\limits_{\theta_j \in \Theta} f_{jk}^- = 7$.

2. Select the minimum acceptable risk, namely: $f_0 = 8 \geq f = 7$.

3. Calculate Bayesian values $B^-(\varphi_k, p_0)$, for those decisions φ_k, which satisfy such condition: $f_0 \geq \max\limits_{\theta_j \in \Theta} f_{jk}$.

In this task, this will be a solution φ_1, φ_2, φ_5. So, $B^-(\varphi_1, p_0) = 5$, $B^-(\varphi_2, p_0) = 4{,}7$, $B^-(\varphi_5, p_0) = 4{,}4$.

The lowest value of the Bayesian criterion corresponds to the decision φ_1, therefore, this choice can be considered optimal.

SELF-STUDY

Questions for assessment and self-assessment

1. What is an informational situation as the basis for making decisions?
2. Give the definition to a decision-making criterion.
3. What elements describe a situation of making decisions?
4. Is the difference between the estimated functionals with a positive and negative ingredient?
5. Name the stages of the decision-making process.
6. What features characterize a decision-making situation in risky conditions?
7. Describe the criteria which are applied in risky situations.
8. When is it appropriate to apply the Bayesian criterion?
9. Under what conditions should decisions be made on the basis of the minimum dispersion criterion? What are the features of its application?
10. In what situations does a modal criterion apply?
11. Give the definition of a Bayesian set.

12. What are the properties of Bayesian sets?

13. What purpose are Bayesian sets used for?

14. What methods of building Bayesian sets do you know?

15. What is the geometric method for constructing Bayesian sets? When it can be used?

16. How can the essence of the functional method of constructing Bayesian sets be defined? When it can be used?

17. Formulate the algorithm of the method of variation of the reference point for constructing Bayesian sets?

18. What criteria can be used in conditions of complete uncertainty of decision-making situation?

19. What is the meaning of the Bernoulli-Laplace criterion?

20. Describe the integral decision criteria (maximal measure of Bayesian sets, integral Bayes value of the estimated functional, integral potential)

21. What criteria should be used in conditions of antagonistic behaviour of the environment?

22. What are disadvantages and advantages of the Savage and Wald criteria.

23. In what situation it is expedient to apply the criteria of Hurwitz and Hodges-Lehmann? Describe the meaning of each of these criteria.

Hands-on practice

Task A

1. Find optimal solutions in risky conditions using the criteria of the first informational situation, when the decision-making situation is described as follows:

a)

p		φ_1	φ_2	φ_3	φ_4
0,1	θ_1	1	5	3	0
0,3	θ_2	2	2	2	4
0,5	θ_3	4	2	3	6
0,1	θ_4	0	4	3	1

b)

p		φ_1	φ_2	φ_3	φ_4
0,2	θ_1	1	5	3	4
0,3	θ_2	3	2	2	4
0,4	θ_3	4	3	3	0
0,1	θ_4	1	4	3	1

c)

p		φ_1	φ_2	φ_3	φ_4
0,1	θ_1	5	4	0	0
0,5	θ_2	1	3	7	4
0,3	θ_3	0	5	2	6
0,1	θ_4	0	2	3	4

d)

p		φ_1	φ_2	φ_3	φ_4
0,1	θ_1	2	3	0	1
0,3	θ_2	3	2	2	4
0,3	θ_3	2	2	4	6
0,3	θ_4	2	4	3	1

e)

p		φ_1	φ_2	φ_3	φ_4
0,4	θ_1	2	5	3	1
0,3	θ_2	1	3	2	4
0,2	θ_3	4	2	3	5
0,1	θ_4	5	0	3	2

f)

p		φ_1	φ_2	φ_3	φ_4
0,1	θ_1	2	5	3	0
0,3	θ_2	5	0	2	4
0,5	θ_3	4	2	3	6
0,1	θ_4	7	4	3	3

2. Construct the Bayesian set using geometric method, when the decision-making situation is given by one of the following matrices:

a)

	φ_1	φ_2	φ_3	φ_4
θ_1	1	5	2	5
θ_2	4	0	3	2

b)

	φ_1	φ_2	φ_3	φ_4
θ_1	1	5	2	5
θ_2	4	0	3	2

c)

	φ_1	φ_2	φ_3	φ_4
θ_1	2	5	2	4
θ_2	2	3	7	2

d)

	φ_1	φ_2	φ_3	φ_4
θ_1	0	5	2	6
θ_2	4	3	3	2

3. Build the Bayesian set with a geometric method and a reference point method, if the decision-making situation is specified by one of the following matrices:

a)

p_0		φ_1	φ_2	φ_3
0,1	θ_1	1	5	7
0,4	θ_2	7	2	1
0,5	θ_3	4	1	3

b)

p_0		φ_1	φ_2	φ_3
0,4	θ_1	12	4	5
0,3	θ_2	10	6	12
0,3	θ_3	7	10	7

c)

p_0		φ_1	φ_2	φ_3
0,2	θ_1	1	0	3
0,5	θ_2	1	4	1
0,3	θ_3	5	1	1

d)

p_0		φ_1	φ_2	φ_3
0,2	θ_1	11	15	20
0,5	θ_2	17	3	14
0,3	θ_3	11	5	7

e)

p_0		φ_1	φ_2	φ_3
0,4	θ_1	3	1	3
0,2	θ_2	2	2	2
0,4	θ_3	1	4	5

f)

p_0		φ_1	φ_2	φ_3
0,1	θ_1	2	0	1
0,1	θ_2	1	2	3
0,8	θ_3	1	4	0

4. Choose an optimal solution using the criteria of the maximum measure of Bayesian sets, the maximum integral Bayes value, and the maximum of the integral potential for the decision-making situation from task 2.

5. Find the optimal solution using Bernoulli-Laplace's criterion when the decision-making situation is given as follows:

a)

	φ_1	φ_2	φ_3	φ_4
θ_1	2	5	0	0
θ_2	2	2	2	4
θ_3	2	0	3	6
θ_4	2	4	3	1

b)

	φ_1	φ_2	φ_3	φ_4
θ_1	1	7	3	0
θ_2	2	2	5	4
θ_3	4	0	3	6
θ_4	1	4	3	1

6. Accept the optimal solution using the Hurwitz criterion in the following conditions:

a)

	φ_1	φ_2	φ_3	φ_4
θ_1	1	2	3	0
θ_2	5	2	2	4
θ_3	4	4	3	6
θ_4	0	4	3	1

$\lambda = 0{,}7$

b)

	φ_1	φ_2	φ_3	φ_4
θ_1	1	5	3	2
θ_2	1	3	2	4
θ_3	4	2	4	6
θ_4	7	4	3	1

$\lambda = 0{,}5$

7. Build sets of Hurwitz for the decision-making situation of Task 6.

8. Find the optimal solution using the Hodges-Lehman criterion, in the context of such a decision-making situation:

a)

p_0	F^+	φ_1	φ_2	φ_3	φ_4
0,1	θ_1	1	5	3	2
0,3	θ_2	2	6	2	4
0,5	θ_3	4	2	2	6
0,1	θ_4	1	4	3	1

b)

p_0	F^-	φ_1	φ_2	φ_3	φ_4
0,35	θ_1	1	5	3	0
0,15	θ_2	3	2	3	4
0,4	θ_3	4	2	3	6
0,1	θ_4	0	4	3	1

Task B

Submit the tasks described below as a mathematical model of the decision-making situation and solve them.

1. The tourist agency has 10 cars for transportation of customers. It plans its work for the next year. It can: leave the number of cars unchanged, take an additional number of cars for rent or lend several of them for renting. The cost of maintaining each car, the profit it brings when working with full load, the cost of leasing additional cars and the profit from lending a car (in conventional units) is shown in Table 6.3. The actual need for cars is random and depends on many unknown factors. Possible states of the environment are described as follows:

- a slight increase in the number of tourists is expected;
- the number of tourists will increase significantly;
- the number of customers will be constant;
- the number of customers will be reduced significantly;
- there will be almost no customers.

Table 6.3

The cost of maintaining one car	5
Costs for renting one car	15
Profit from the work of one car	5
Profit from renting a car	5

2. Igor is a dedicated football fan. He periodically plays on the totalizator, betting on the results of a match. Now he needs to decide what amount of money from possible options: $F = \{100, 200, 300, 400\}$ to bet, taking into account the five possible scenarios for the match results:

Q_1 – victory with a probability of 0,3;

Q_2 – defeat with a probability of 0,2;

Q_3 – victory over extra time – 0,1;

Q_4 – victory according to the results of puncture penalty – 0,1;

Q_5 – Draw with a probability of 0,3.

Describe the situation of decision-making and recommend Igor a strategy of his behaviour, based on the following assumptions about his arguments:

a) confident in his assessment of the results of the match;

b) not sure in his assessment of the results of the match;

c) does not want to risk;

d) interested in the greatest winnings.

3. According to the data of statistical observations, the data about the dependence of the relative yield of agricultural crops on the initial soil moisture and the selected irrigation program were obtained and summarized in Table 6.4. The farmer needs to make a selection of the irrigation program in case the probability of the moisture content is known.

Table 6.4

Humidity of the soil, %	Irrigation program				Probability of humidity values
	P1	P2	P3	P4	
10	0,13	0,40	1,00	1,00	0,03
20	0,28	0,40	1,00	0,98	0,08
30	0,37	0,53	1,00	0,96	0,1
40	0,60	0,61	0,96	0,90	0,12
50	0,58	0,74	0,94	0,86	0,18
60	0,62	0,86	0,88	0,75	0,14
70	0,76	1,00	0,61	0,50	0,1
80	0,86	1,00	0,53	0,40	0,12
90	0,90	1,00	0,40	0,32	0,08
100	0,99	0,72	0,40	0,28	0,05

4. The law enforcement guard noticed two suspicious young men who are brazenly behaving in a crowded city, drawing attention to themselves, but not committing unlawful actions. His job description provides for such a variety of alternative actions in the described situation:

φ_1 – to stop the actions of potential offenders without waiting for help:

φ_2 – call for help and wait for it to neutralize potential offenders;

φ_3 – wait for the young people to commit an unlawful act in order to have a reason to apply force to them;

φ_4 – make remarks to the young people;

φ_5 – do not pay attention to these people.

The set of possible states of the environment is estimated as follows:

θ_1 – young people are innocent, just in good spirits and having a rest;

θ_2 – young people are poorly controlled potential criminals who stumble when they do not have the opposite;

θ_3 – young people are provokers, their goal is to seize a law-enforcers' piece of weapons;

θ_4 – young people – re-dressed quality assurance officers from the department of internal investigations.

The decision-making situation should take into account both the potential danger to the law-enforcers and the danger of the situation and its consequences for citizens, whose peace is the purpose of law-enforces activities.

Describe the situation of decision-making and recommend the law-enforcement strategy of behaviour based on different assumptions.

5. In anticipation of the sowing season, a farmer must choose one of the following alternatives to their activity:

φ_1 – grow corn;

φ_2 – grow wheat;

φ_3 – grow beans;

φ_4 – use land under cattle grazing.

The magnitude of the costs associated with these opportunities (see Table 6.5) depends on the amount of precipitation that can be divided into four categories:

θ_1 – heavy precipitation;

θ_2 – moderate precipitation;

θ_3 – slight precipitation;

θ_4 – arid weather.

Table 6.5

	φ_1	φ_2	φ_3	φ_4
θ_1	–20	40	–50	12
θ_2	60	50	100	15
θ_3	30	35	45	15
Θ_4	–5	0	–10	10

Give advice to the farmer in choosing a solution under the following conditions:

a) the probability of rainfall is known, namely: p = (0,3; 0,4; 0,2; 0,1).

b) the farmer believes that he is not always happy;

c) the farmer does not trust the weather forecast;

d) the farmer considers the weather forecast to be quite accurate, but he wants to provide certain guarantees of successful activity in the event that the forecast does not materialize.

6. The company plans to produce new products. Experts of the research department are confident that new products will be in high demand, and therefore insist on its immediate introduction into production without an advertising campaign in markets. The marketing department evaluates the state of things in a different way and proposes to hold an intensive advertising campaign, which will cost the company 100 000 UAH, and if successful, will bring 950 000 UAH. In case of failure of the advertising campaign, the annual profit is estimated at only 200 000 UAH. If the advertising campaign is not carried out at all, then the expected profit is estimated at 400 000 UAH, when new products will appeal to consumers and 200 000 when they remain indifferent to new products.

Give recommendations to the director of the company choosing a solution based on the following conditions:

a) an advertising campaign with a probability of 0,8 is expected to be successful;

b) the reaction of buyers to new products can not be predicted since such goods have not yet been on the market;

c) the director trust more in the marketing department;

d) the director considers the forecast to be rather precise, but seeks to secure certain guarantees of the company's success.

Task C.

1. Formulate a task that is modeled by the first informational situation. It is necessary to determine:

– a set of states of the environment;

– a set of possible decisions of the management/decision-making body;

– functional, by which one can evaluate the quality of a solution in a particular situation;

– a priori probabilities on the set of states of the environment.

The compiled task is to apply as a mathematical model of the decision-making situation and to solve it.

2. Solve *Task C-1* taking into account the antagonistic behaviour of the environment, its partial or complete uncertainty.

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English-Ukrainian Glossary of Terms

A

Aditivity	Адитивність
Alternative	**Альтернатива**
– (clearly) undominated	– (чітко) недомінована
– effective	– ефективна
– optimal to Pareto	– оптимальна за Парето
– optimal to Slayter	– оптимальна за Слейтером
– undominated	– недомінована
– unsurpassed for a set of purposes	– непокращувана за множиною цілей
– weakly effective	– слабко ефективна
– weakly optimal to Pareto	– слабко оптимальна за Парето

B

Bayes value of the estimated functional	Байєсове значення оцінного функціонала
Bernoulli – Laplace criterion	Бернуллі – Лапласа критерій
Binary relation	Бінарне відношення

C

Complementary (of)	**Доповнення**
– fuzzy relation	– нечіткого відношення
– fuzzy set	– нечіткої множини
– relation	– відношення
Composition of relations	Композиція відношень
Concentration	Концентрування
Constraints	Обмеження
Convex combination of sets	Опукла комбінація множин
Criterion	**Критерій**
Bayesian –	– Байєса
Bernoulli – Laplace –	– Бернуллі – Лапласа
Combined –	– комбінований
Decision-making –	– прийняття рішень
Hodges – Lehman –	– Ходжеса – Лемана
Hurwitz –	– Гурвіца

– of the maximum integral Bayes value of the estimated functional	– максимального інтегрального байєсового значення оцінного функціонала
– of the maximum integral potential	– максимального інтегрального потенціалу
– of the maximum measure of Bayesian sets	– максимальної міри байєсових множин
– of maximizing the distribution of probability of the estimated functional	– максимізації ймовірності розподілу оцінного функціонала
Partial –	– частковий
Savage –	– Савіджа
– of the minimal entropy of the mathematical expectation of the estimated functional	– мінімальної ентропії математичного сподівання оцінного функціонала
– of the minimum dispersion of the estimated functional	– мінімуму дисперсії оцінного функціонала
Wald's –	– Вальда
Criterion space	Критерійний простір

D

Decision	**Рішення**
– Bayesian	– байєсове
– limited	– обмежене
– conditional	– умовне
– minimum	– мінімальне
Decision maker (DM), decision-making person (DMP)	Особа, що приймає рішення (ОПР)
Decision-making	**Прийняття рішень**
– in risky conditions	– в умовах ризику
Degree of membership	Ступінь належності
Dilatation	Розтягування

E

Element(s)	**Елементи**
– non-comparable	– непорівнянні

F

Function (of)	**Функція**
– membership	– належності
– characteristic	– характеристична
– choice	– вибору
– growing by the relation *R*	– зростаюча за відношенням *R*
– utility	– корисності
– equivalent	– еквівалентна
Functional	Функціонал
Fuzzy	Нечіткий, невизначений
Fuzzy goal/purpose	Нечітка мета
Fuzzy index	**Індекс нечіткості**
– – quadratic	– – квадратичний
– – linear	– – лінійний
Fuzzy mathematical programming	Нечітке математичне програмування
Fuzzy purpose	Нечітка мета/ціль
Fuzzy relation	**Нечітке відношення**
– antisymmetric	– антисиметричне
– antireflexive, irreflexive	– антирефлексивне
– asymmetric	– асиметричне
– complete	– повне
– converse	– обернене
– empty	– порожнє
– of equality	– рівності
– of indifference	– байдужості
– of non-substantial preference	– нестрогої переваги
– (of) pre-order	– передпорядку
– of quasi equivalence	– квазіеквівалентності
– reflexive	– рефлексивне
– of similarity	– подібності, схожості
– (of) strict order	– строгого порядку
– of strong preference	– строгої переваги
– strongly linear	– сильно лінійне
– symmetric	– симетричне
– transitive	– транзитивне
– of uniformity	– однаковості
– weakly linear	– слабко лінійне
– λ-line	– λ-лінійне

Fuzzy set	**Нечітка множина**
– – normal	– – нормальна
– – subnormal	– – субнормальна
Fuzzy solution	**Нечіткий розв’язок**
– – ε-optimal	– – ε-оптимальний

I

Image of the set	**Образ множини**
– – with the fuzzy mapping	– – при нечіткому відображенні
– – with the classical mapping	– – при звичайному відображенні
Informational situation (of)	**Інформаційна ситуація**
– – decision-making in fuzzy conditions/terms	– – прийняття рішень у нечітких умовах
Ingredient	**Інгредієнт**
– negative	– від’ємний
– positive	– додатний
Integral potential	Інтегральний потенціал
Integral Bayes value	Інтегральне байєсове значення
Intersection (of)	**Перетин**
– (fuzzy) relations	– (нечітких) відношень
– fuzzy sets (subsets)	– нечітких множин (підмножин)
– relations	– відношень

L

Linear order	Лінійний порядок

M

Map, mapping	Відображення
Method(s) of	**Метод(и)**
– criteria normalization	– нормалізації критеріїв
– decision-making	– прийняття рішень
– flexible priority consideration	– врахування гнучкого пріоритету
– restrictions	– обмежень
– successive concessions	– послідовних поступок
– summary to the generalized criterion [convolution(s)]	– зведення до узагальненого критерію (згортки)
– the main criterion	– головного критерію

– the maximum integral Bayesian value of the estimated functional	– максимального інтегрального байєсового значення оцінного функціонала
– tough priority	– жорсткого пріоритету
– Wald's	– Вальда
Model	**Модель**
– deterministic	– детермінована
– dynamic	– динамічна
– static	– статична
– stochastic	– стохастична

N

Normalising criteria, Normalization of criteria	**Нормалізація критеріїв**

O

Order	**Порядок**
– linear	– лінійний
– not strict	– нестрогий
– partial	– частковий
– strict	– строгий

P

Preference	Перевага
Principle of	**Принцип**
– a fair acquiescence with a priority	– справедливої поступки з пріоритетом
– an absolute acquiescence	– абсолютної поступки
– an alignment of quality	– вирівнювання якості
– equality	– рівності
– generalization	– узагальнення
– insufficient grounds	– недостатньої підстави
– maximin	– максиміну
– maximizing probability of achieving the ideal quality	– максимізації ймовірності досягнення ідеальної якості
– maximizing the weighted sum of the criteria	– максимізації зваженої суми критеріїв

– quasiveness	– квазірівності
– relative act	– відносної поступки
– the best uniformity	– найкращої рівномірності
– the main criterion	– головного критерію
– the slightest damage	– найменшої шкоди
– uniformity (maximin)	– рівномірності (максиміну)
– uniformity with priority	– рівномірності з пріоритетом
Probability distribution	**Розподіл імовірності**
– objective	– об'єктивний
– subjective	– суб'єктивний
Problem (of)	**Задача**
– fuzzy mathematical programming	– нечіткого математичного програмування
– mathematical programming with fuzzy constraints	– математичного програмування з нечіткими обмеженнями
– multi-criteria optimization	– багатокритерійної оптимізації
Product (of)	**Добуток**
– maximinium	–максимінний
– relations	– відношень

R

Relation	**Відношення**
– acyclic	– ациклічне
– antireflexive, irreflexive	– антирефлексивне
– anti-symmetric	– антисиметричне
– asymmetric	– асиметричне
– of differences	– відмінності
– diagonal	– діагональне
– of domination	– домінування
– of equivalence	– еквівалентності
– linear	– лінійне
– negative transitive	– від'ємно транзитивне
– of non-strict order	– нестрогого порядку
– of non-substantial preference	– нестрогої переваги
– of indifference	– байдужості
Relation cut(s)	**Розріз відношення**
– lower cut or undercut	– нижній
– upper cut	– верхній

S

Set	**Множина**
– Bayeseian	– байєсова
– Hurwitz	– Гурвіца
– internally stable	– внутрішньо стійка
– usual the closest to the fuzzy	– звичайна найближча до нечіткої
– external stable	– зовнішньо стійка
– nondominated alternatives	– недомінованих альтернатив
– usual the closest to the fuzzy	– звичайна найближча до нечіткої
Fuzzy set	**Нечітка множина**
– normal	– – нормальна
– subnormal	– – субнормальна
Set partition (in)	**Розбиття множини**
Solving, solution	**Розв'язок**
– fuzzy	– нечіткий
– maximizing	– максимізувальний
Strategy	**Стратегія**
Support (of)	**Носій**
– fuzzy relation	– нечіткого відношення
– fuzzy set (subset)	– нечіткої множини (підмножини)

T

Transitive closure of fuzzy relation	Транзитивне замикання нечіткого відношення
Transitivity	**Транзитивність**
– maximin	– максимінна
– maxmultiplicative	– максмультиплікативна
– minimax	– мінімаксна

U

Union of	**Об'єднання**
– relations	– відношень
– fuzzy relations	– нечітких відношень
Utility	Корисність

Ukrainian-English Glossary of Terms

А	
Адитивність	**Aditivity**
Альтернатива	**Alternative**
– ефективна	– effective
– недомінована	– undominated
– непокращувана за множиною цілей	– unsurpassed for a set of purposes
– оптимальна за Парето	– optimal to Pareto
– оптимальна за Слейтером	– optimal to Slayter
– слабко ефективна	– weakly effective
– слабко оптимальна за Парето	– weakly optimal to Pareto
– чітко недомінована	– clearly undominated
Б	
Багатокритерійна задача	Multi-criteria problem
Байєсове значення оцінного функціонала	Bayes value of the estimated functional
Бернуллі – Лапласа критерій	Bernoulli – Laplace criterion
Бінарне відношення	Binary relation
В	
Вальда критерій	Wald's criterion
Варіант	Variant
Вектор ідеальний	ideal vector
Відношення	**Relation**
– антирефлексивне	– antireflexive, irreflexive
– антисиметричне	– antisymmetric
– асиметричне	– asymmetric
– ациклічне	– acyclic
– байдужості	– of indifference
– від'ємно транзитивне	– negative transitive
– відмінності	– of differences
– діагональне	– diagonal
– домінування	– of domination
– еквівалентності	– of equivalence

– лінійне	– linear
– нестрогого порядку	– (of) non-strict order
– нестрогої переваги	– (of) non-substantial benefits
– нечітке	– fuzzy
– обернене	– converse
– однаковості	– (of) uniformity
– передпорядку	– (of) pre-order
– повне	– complete
– подібності	– of similarity
– порожнє	– empty
– рефлексивне	– reflexive
– рівності	– of equality
– сильно лінійне	– strongly linear
– симетричне	– symmetric
– слабко лінійне	– weakly linear
– строгого порядку	– (of) strict order
– строгої переваги	– of strong advantage
– схожості	– of similarity
– λ-лінійне	– λ-line
– транзитивне	– transitive
Відображення нечітке	fuzzy mapping
Відстань	**Distance**
– евклідова (квадратична)	– Euclidean (quadratic)
– відносна	– relative
– Хеммінга	– Hamming
–узагальнена	– generalized

Д

Добуток	**Product (of)**
– відношень	– relations
– декартів	– Cartesian
– нечітких відношень	– fuzzy relations
– максимінний	– maximinium
– нечітких відношень максмультиплікативний	– fuzzy relations maxmultiplicative
– нечітких відношень мінімаксний	– fuzzy relations minimax

Доповнення	**Complement(ary) (of)**
– відношення	– relation
– нечіткого відношення	– fuzzy relation
– нечіткої множини	– fuzzy set
Е	
Еквівалентність	**Equivalence (of)**
– множин	– sets
– рішень	– solutions
Елемент	**Element**
– максимальний	– maximum, maximal
– мінімальний	– minimum, minimal
– найгірший	– the worst
– найкращий	– the best
Елементи непорівнянні	Non-comparable elements
З	
Задача	**Problem (of)**
– вибору	– choice
– нечіткого математичного програмування	– fuzzy mathematical programming
– прийняття рішень у нечітких умовах	– decision-making in fuzzy terms
– багатокритерійної оптимізації	– multi-criteria optimization
– математичного програмування з нечіткими обмеженнями	– mathematical programming with fuzzy constraints
Звуження відношень	Narrowing relationships
Значення функції нечітке	Fuzzy function value
І	
Інгредієнт	**Ingredient**
– від’ємний	– negative
– додатний	– positive
Індекс нечіткості	**Fuzzy index**
– квадратичний	– quadratic
– лінійний	– linear
Інтегральний потенціал	Integral potential
Інтегральне байєсове значення	Integral Bayes Value

Інформаційна ситуація	Informational situation
К	
Композиція відношень	Composition of relations
Концентрування	Concentration
Корисність	Utility
Крива Гурвіца	Curve of Hurwitz
Критерій	**Criterion**
– Байєса	– Bayes
– Бернуллі – Лапласа	– Bernoulli – Laplace
– Вальда	– Wald's
– Гурвіца	– Hurwitz
– комбінований	– combined
– максимального інтегрального байєсового значення оцінного функціонала	– the maximum integral Bayesian value of the estimated functional
– максимального інтегрального потенціалу	– the maximum integral potential
– максимальної міри байєсових множин	– the maximum degree of Bayesian sets
– максимізації ймовірності розподілу оцінного функціонала	– maximizing the distribution of probability of the estimated functional
– мінімальної ентропії математичного сподівання оцінного функціонала	– the minimal entropy of the mathematical expectation of the estimated functional
– мінімуму дисперсії оцінного функціонала	– the minimum dispersion of the estimated functional
– прийняття рішень	– decision-making
– Савіджа	– Savidge
– Ходжеса – Лемана	– Hodges – Lehman
– частковий	– partial
Критерійний простір	Criterion space
Л	
Лінійний порядок	Linear order

М

Мета нечітка	Fuzzy purpose
Метод(и)	**Method(s) (of)**
– врахування гнучкого пріоритету	– flexible priority consideration
– головного критерію	– the main criterion
– жорсткого пріоритету	– tough priority
– зведення до узагальненого критерію (згортки)	– summary to the generalized criterion [convolution(s)]
– нормалізації критеріїв	– criteria normalization
– обмежень	– restrictions
– послідовних поступок	– successive concessions
Множина	**Set**
– байєсова	– Bayeseian
– Гурвіца	– Hurwitz
– внутрішньо стійка	– internally stable
– звичайна найближча до нечіткої	– usual the closest to the fuzzy
– зовнішньо стійка	– outboard stand
– недомінованих альтернатив	– undocumented alternatives
Множина нечітка	**Fuzzy set**
– – нормальна	– – normal
– – субнормальна	– – subnormal
Модель	**Model**
– детермінована	– deterministic
– динамічна	– dynamic
– статична	– static
– стохастична	– stochastic

Н

Недомінована альтернатива	Undominating alternative
Нечітка мета	Fuzzy goal
Нечітке відношення	**Fuzzy relation of**
– – байдужості	– – indifference
– – квазіеквівалентності	– – quasiequivalence
– – нестрогої переваги	– – non-substantial preference
– – однаковості	– – homogeneity
– – строгої переваги	– – a strong preference

Нечітке математичне програмування (НМП)	Fuzzy mathematical programming (FMP)
Нечіткий розв'язок	**Fuzzy solution**
– – ε-оптимальний	– – ε-optimal
Нормалізація критеріїв	Normalising criteria, Normalization of criteria
Носій	**Support (of)**
– нечіткого відношення	– fuzzy relation
– нечіткої множини (підмножини)	– fuzzy set (subset)
О	
Об'єднання	**Union (of)**
– відношень	– relations
– нечітких множин	– fuzzy sets
– нечітких відношень	– fuzzy relations
Обмеження	Constraint(s)
Образ множини	**Image of the set**
– – при нечіткому відображенні	– – with the fuzzy mapping
– – при звичайному відображенні	– – with the classical mapping
Опукла комбінація множин	Convex combination of sets
Особа, що приймає рішення (ОПР)	Decision maker (DM), Decision-making person (DMP)
П	
Перетин	**Intersection(of)**
– відношень	– relations
– нечітких множин (підмножин)	– fuzzy sets (subsets)
– нечітких відношень	– fuzzy relations
Підмножина λ-рівня	Subset of λ-level
План	Plan
Поверхня байєсова	Bayesian surface
Показник Гурвіца	Hurwitz index
Порядок	**Order**
– лінійний	– linear
– нестрогий	– not strict
– строгий	– strict
– частковий	– partial

Прийняття рішень	**Decision-making**
– – в умовах ризику	– in risky conditions
Принцип	**Principle (of)**
– абсолютної поступки	– an absolute assignment
– вирівнювання якості	– an alignment of quality
– відносної поступки	– relative assignment
– головного критерію	– the main criterion
– квазірівності	– quasiveness
– максимізації ймовірності досягнення ідеальної якості	– maximizing probability of achieving the ideal quality
– максимізації зваженої суми критеріїв	– maximizing the weighted sum of the criteria
– максиміну	– maximin
– найкращої рівномірності	– the best uniformity
– найменшої шкоди	– the slightest damage
– недостатньої підстави	– insufficient grounds
– рівномірності з пріоритетом	– uniformity with priority
– рівномірності (максиміну)	– uniformity (maximin)
– рівності	– equality
– справедливої поступки з пріоритетом	– a fair assignment with a priority
– узагальнення	– generalization
Програмування цільове	Target Programming
Прообраз нечіткої множини	Prototype of fuzzy set

Р

Різниця нечітких множин	**Difference between fuzzy sets**
Рішення	**Decision**
– байєсове	– Bayesian
– обмежене	– limited
– умовне	– conditional
Розбиття множини	Set partition
Розв’язок	**Solving, solution**
– нечіткий	– fuzzy
– максимізувальний	– maximizing
Розподіл імовірності	**Probability distribution**
– – об’єктивний	– – objective
– – суб’єктивний	– – subjective

Розріз відношення	**Relation cut(s)**
– – верхній	– – upper cut
– – нижній	– – lower cut or undercut
Розтягування	**Dilatation**

С

Ситуація прийняття рішень	Decision-making situation
Ступінь належності	Degree of membership
Стратегія	Strategy

Т

Транзитивне замикання нечіткого відношення	Transient closure of fuzzy relation
Транзитивність	**Transitivity**
– максимінна	– maximin
– максмультиплікативна	– maxmultiplicative
– мінімаксна	– minimax

Ф

Функція	**Function of**
– вибору	– choice
– зростаюча за відношенням R	– growing by the ratio R
– корисності	– utility
– належності	– affiliation
– характеристична	– characteristic
Функції еквівалентні	Equivalent functions

Ч

Частковий порядок	Partial order

SUBJECT INDEX

Навчальне видання

Ус Світлана Альбертівна
Коряшкіна Лариса Сергіївна
Зуєнок Ірина Іванівна

МОДЕЛІ Й МЕТОДИ ПРИЙНЯТТЯ РІШЕНЬ

Навчальний посібник

(Англійською мовою)

Видано в редакції авторів

Підп. до друку 10.04.2019. Формат 30х42/4.
Папір офсет. Ризографія. Ум. друк. арк. 17,4.
Обл.-вид. арк. 23,0. Тираж 100 пр. Зам. №

Підготовлено до друку та видрукувано
в Національному технічному університеті
«Дніпровська політехніка»

Свідоцтво про внесення до Державного реєстру ДК № 1842 від 11.06.2004 р.
49005, м. Дніпро, просп. Д. Яворницького, 19.