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**THEORETICAL MECHANICS  
DYNAMICS**

Tutorial

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**МІНІСТЕРСТВО ОСВІТИ І НАУКИ, МОЛОДІ ТА СПОРТУ УКРАЇНИ  
ДЕРЖАВНИЙ ВИЩИЙ НАВЧАЛЬНИЙ ЗАКЛАД  
«НАЦІОНАЛЬНИЙ ГІРНИЧИЙ УНІВЕРСИТЕТ»**



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Contents of this tutorial correspond to the bachelors' educational and professional program in Electromechanics, particularly, to the subject of Theoretical Mechanics. Notions of particle and system dynamics are considered. Examples of practical problem solutions and tasks for self-study training, which define bachelor's knowledge and abilities in Electromechanics, are represented.

Зміст навчального посібника відповідає освітньо-професійній програмі підготовки бакалаврів з напряму «Електромеханіка», зокрема – дисципліні «Теоретична механіка». Розглянуто питання динаміки матеріальної точки і механічної системи. Наведено приклади розв'язування практичних задач і сформульовано завдання для самостійної роботи, які окреслюють навички та уміння бакалавра напряму «Електромеханіка».

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## PREFACE

The primary object of Dynamics to be gained by the student is a thorough grasp of fundamental principles. In most cases it is impossible to go beyond this object in the time available for the course. In the preparation of this textbook, the aim has been to present the fundamental principles in as clear and simple a manner as possible, and to enforce them by a sufficient number of illustrative examples.

The study of Dynamics, as presented in this manual, is founded upon a course in Statics and Kinematics. It is assumed, moreover, that the students have already become familiar with the fundamental ideas of force, energy and work through such preliminary courses of General Physics. The mathematical training required for using the book is that usually implied by an elementary knowledge of Differential and Integral Calculus.

In short, this textbook presents the subject of Dynamics in that relation to other subjects which have become established in the curricula of the technical universities. It should be emphasized, however, that the manual includes, for purposes of review, a discussion of the fundamental notions and many problems involving these notions. Attention may be called to the arrangement in the text. This arrangement is founded upon experience in teaching the subject for many years in the National Mining University of Ukraine. This manual is based on a Short Course of Theoretical Mechanics by S.M. Targ (Foreign Languages Publishing House, Moscow) [1] and prepared for foreign students and for those who study some subjects of the curricula in English.

The opinion is sometimes expressed that the needs of different classes of students require essentially different methods of treating the subject. This view, so far as it refers to the fundamental parts of an elementary course of Dynamics, is not shared by the author of this textbook. For all students, the matter of first importance is the clear understanding of fundamental general principles and the ability to apply them. That is why there are included in the text some of problems suggested for independent work on the course. These problems are taken from [2]. They were chosen with an eye to ensure a clear comprehension of the dynamical phenomena, and they embrace all the main methods of Dynamics. In order to assist the students' work the examples of the problem solutions contain the relevant instructions.

The author hopes that this manual may be useful to students of technical specialties interested in advancing their knowledge of Dynamics. If this book is in any degree successful in meeting the needs of students of engineering, it is hoped that it may be of service also to those pursuing the subject for its intrinsic scientific interest or as a preparation for the study of other engineering disciplines.

The author should be greatly obliged to those who may make use of the book if they would point out any defects or obscurities in the text or would offer suggestions for its improvement.

## INTRODUCTION

This text is intended for the first course in the study of Theoretical Mechanics, and its part, Dynamics, usually taken by engineering students in the sophomore or junior year. It is assumed that the student has completed the basic courses in physics, calculus, statics, and kinematics.

The purpose of the study of Dynamics is twofold. First, students must be introduced to the basic ideas and concepts used in the area of Dynamics. This includes a thorough treatment of the basic ideas of mass, acceleration, force, energy, work, mechanical system, measures of mechanical interaction and motion, differential equations of motion, and so on. These ideas are emphasized and kept in focus throughout this text, with careful study of how their combination leads to specific theories about motion of material bodies.

Second, students need ample practice in applying these theories to practical situations. Relatively simple problems are examined in this text to analyze motion. Both of these goals require continual awareness of all the notions that are necessary parts of Dynamics, to understand and avoid situations where application of theory is unwarranted.

A traditionally difficult aspect of developing a Dynamics text has been in striking a balance between theory and the many practical applications that are important to students who will go on to use the knowledge in actual practice.

A major goal is to keep the basic ideas clearly in focus when developing theory or applying the results of the theory to actual situations. In experience, a careful separate treatment of each of the basic ideas provides an excellent framework for the study of elementary theories in Dynamics.

Section 1 presents an overview of the typical areas of application of Dynamics. The three basic ideas of Dynamics, force, mass, and acceleration, are introduced. The laws and problems of Dynamics are formulated.

The main goal of Dynamics can be stated roughly as follows. Given the loads applied to the body, what is the law of its motion? Discussions in Section 2 involve using differential equations of particle motion to determine its law of motion.

Section 3 is devoted to the study of characteristic features of motion, i.e., vibrations. The main notions and conclusions are discussed in detail. The phenomenon of resonance is considered.

Section 4 has auxiliary character for the study of System Dynamics. Some of important concepts are introduced and discussed.

So called general theorems of Dynamics for particle and mechanical system are covered in Section 5. In order to consider these theorems, many concepts are introduced. Using theorems solves the basic problem of Dynamics without integrating differential equations of motion in many practical applications.

Some elements of Analytical Mechanics are considered in Section 6. This Section studies principles of Dynamics which represent general approach to the dynamic and static problems.



Section 7 deals with Lagrangian techniques for developing differential equations of motion for mechanical system. The advantages' of Lagrange's equations are demonstrated in detail. Very likely, this Section is the most important from the practical point of view.

All Sections except Section 4 contain examples of problems solution.

There are problems for self-instruction with examples of their solution in Section 8. These, more challenging problems, are intended to assist the students in understanding and applying the basic ideas.

# 1. LAWS AND PROBLEMS OF DYNAMICS

## 1.1. Basic Concepts and Definitions

Dynamics is that section of mechanics which treats of the laws of motion of material bodies subjected to the action of forces.

The motion of bodies from a purely geometrical point of view is discussed in kinematics. Unlike kinematics, in dynamics the motion of bodies is investigated in connection with the acting forces and the inertia of the material bodies themselves.

The concept of force as a quantity characterizing the measure of mechanical interaction of material bodies is introduced in the course of statics. But in statics we treat all forces as constant without considering the possibility of their changing with time. In real systems, though, alongside of constant forces a body is often subjected to the action of variable forces whose magnitudes and directions change when the body moves. Variable forces may be both applied (active) forces and the reactions of constraints.

Experience shows that variable forces may depend in some specific ways on time, on the position of a body, or on its velocity (examples of dependence on time are furnished by the tractive force of an electric locomotive whose rheostat is gradually switched on or off, or the force causing the vibration of a foundation of a motor with a poorly centered shaft; the Newtonian force of gravitation or the elastic force of a spring depend on the position of a body; the resistance experienced by a body moving through air or water depends on the velocity). In dynamics we shall deal with such forces alongside of constant forces. The laws for the composition and resolution of variable forces are the same as for constant forces.

The concept of inertia of bodies arises when we compare the results of the action of an identical force on different material bodies. Experience shows that if the same force is applied to two different bodies initially at rest and free from any other actions, in the most general case the bodies will travel different distances and acquire different velocities in the same interval of time.

Inertia is the property of material bodies to resist a change in their velocity under the action of applied forces. If, for example, the velocity of one body changes slower than that of another body subjected to the same force, the former is said to have greater inertia, and vice-versa. The inertia of a body depends on the amount of matter it contains.

The quantitative measure of the inertia of body, which depends on the quantity of matter in the body, is called the mass of that body. In mechanics mass  $m$  is treated as a scalar quantity which is positive and constant for body. The measurement of mass will be discussed in the following article.

In the most general case the motion of a body depends not only on its aggregate mass and the applied forces, the nature of motion may also depend on the dimensions

of the body and the mutual position of its particles (i.e., on the distribution of its mass).

In the initial course of dynamics, in order to neglect the influence of the dimensions and the distribution of the mass of a body, the concept of a material point, or particle, is introduced.

A particle is a material body (a body possessing mass) the size of which can be neglected in investigating its motion.

Actually any body can be treated as a particle when the distances traveled by its points are very great as compared with the size of the body itself. Furthermore, as will be shown in the dynamics of systems, a body in translational motion can always be considered as a particle of mass equal to the mass of the whole body.

Finally, the parts into which we shall mentally divide bodies in analyzing any of their dynamic characteristics can also be treated as material points.

Obviously, the investigation of the motion of a single particle should precede the investigation of systems of particles, and in particular of rigid bodies. Accordingly, the course of dynamics is conventionally subdivided into particle dynamics and the dynamics of systems of particles.

## 1.2. The Laws of Dynamics

The study of dynamics is based on a number of laws generalizing the results of a wide range of experiments and observations of the motions of bodies, i.e., laws that have been verified in the long course of human history.

*The First Law (the Inertia Law): a particle free from any external influences continues in its state of rest, or of uniform rectilinear motion, except in so far as it is compelled to change that state by impressed forces.*

The motion of a body not subjected to any force is called motion under no forces, or inertial motion.

The inertia law states one of the basic properties of matter: that of being always in motion. It establishes the equivalence, for material bodies, of the states of rest and of motion under no forces.

A frame of reference for which the inertia law is valid is called an inertial system (or, conventionally, a fixed system). Experience shows that, for our solar system, an inertial frame of reference has its origin in the center of the sun and its axes are pointed towards the so-called "fixed" stars. In solving most engineering problems a sufficient degree of accuracy is obtained by assuming any frame of reference connected with the earth to be an inertial system.

*The Second Law (the Fundamental Law of Dynamics) establishes the mode in which the velocity of a particle changes under the action of a force. It states: the product of the mass of a particle and the acceleration imparted to it by a force is proportional to the acting force; the acceleration takes place in the direction of the force.*

Mathematically this law is expressed by the vector equation:

$$m\mathbf{a} = \mathbf{F}. \quad (1.1)$$

The second law of dynamics, like the first, is valid only for an inertial system. It can be immediately seen from the law that the measure of the inertia of a particle is its mass, since two different particles subjected to the action of the same force receive the same acceleration only if their masses are equal; if their masses are different, the particle with the larger mass (i.e., the more inert one) will receive a smaller acceleration, and vice-versa.

A set of forces acting on a particle can, as we know, be replaced by a single resultant  $\mathbf{R}$  equal to the geometrical sum of those forces. In this case the equation expressing the fundamental law of dynamics acquires the form

$$m\mathbf{a} = \mathbf{R} \text{ or } m\mathbf{a} = \sum \mathbf{F}_k. \quad (1.2)$$

*Measure of mass.* Eq. (1.2) makes it possible to determine the mass of a body if its acceleration in translational motion and the acting force are known. It has been established experimentally that under the action of the force of gravitation  $\mathbf{P}$  all bodies falling to the earth (from a small height and in vacuum) possess the same acceleration  $g$ , this is known as the acceleration of gravity or of free fall. Applying Eq. (1.2) to this motion, we obtain  $m g = P$ , whence

$$m = \frac{P}{g}. \quad (1.3)$$

Thus, the mass of a body is equal to its weight divided by the acceleration of gravity  $g$ .

*The Third Law (the Law of Action and Reaction)* establishes the character of mechanical interaction between material bodies. For two particles it states: *two particles exert on each other forces equal in magnitude and acting in opposite directions along the straight line connecting the two particles.*

It should be noted that the forces of interaction between free particles (or bodies) do not form a balanced system, as they act on different objects.

The third law of dynamics, which establishes the character of interaction of material particles, plays an important part in the dynamics of systems.

### 1.3. The Problems of Dynamics for a Free and a Constrained Particle

The problems of dynamics for a free particle are: 1) knowing the equation of motion of a particle, to determine the force acting on it (the first problem of dynamics); 2) knowing the forces acting on a particle, to determine its equation of motion (the second, or principal, problem of dynamics).

Both problems are solved with the help of Eq. (1.1) or (1.2), which expresses the fundamental law of dynamics, since they give the relation between acceleration, i.e., the quantity characterizing the motion of a particle, and the forces acting on it.

In engineering it is often necessary to investigate constrained motions of a particle, i.e., cases when constraints attached to a particle compel it to move along a given fixed surface or curve.

In such cases we shall use, as in statics, the axiom of constraints, which states that any constrained particle can be treated as a free body detached from its

constraints provided the latter is represented by their reactions  $N$ . Then the fundamental law of dynamics for the constrained motion of a particle takes the form

$$m\mathbf{a} = \sum \mathbf{F}_k^a + \mathbf{N}, \quad (1.4)$$

where  $\mathbf{F}_k^a$  denotes the applied forces acting on the particle.

For constrained motion the first problem of dynamics will usually be: to determine the reactions of the constraints acting on a particle if the motion and applied forces are known. The second (principal) problem of dynamics for such motion will pose two questions: knowing the applied forces, to determine: a) the equation of motion of the particle and b) the reaction of its constraints.

### 1.4. Solution of Problems

*Problem 1.* A balloon of weight  $P$  descends with acceleration  $a$ . What weight (ballast)  $Q$  must be thrown overboard in order to give the balloon an equal upward acceleration?

*Solution.* The forces acting on the falling balloon are its weight  $P$  and the buoyancy force  $F$  (Fig. 1). Hence, from Eq. (1.2)

$$\frac{P}{g}a = P - F.$$

After the ballast has been thrown out (Fig. 1), the weight of the balloon becomes  $P - Q$ , the buoyancy force remaining the same. Hence, taking into account that now the balloon is rising, we have

$$\frac{P - Q}{g}a = F - (P - Q).$$

Eliminating the unknown force  $F$  from the equations, we obtain

$$Q = \frac{2P}{1 + \frac{g}{a}}.$$

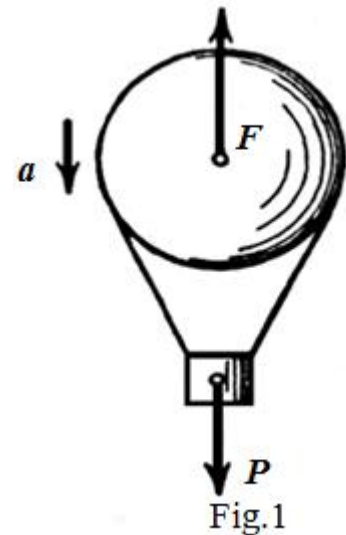


Fig.1

*Problem 2.* A lift of weight  $P$  (Fig. 2) starts ascending with acceleration  $a$ . Determine the tension in the cable.

*Solution.* Considering the lift as a free body, replace the action of the constraint (the cable) by its reaction  $T$ . From Eq. (1.4) we obtain

$$\frac{P}{g}a = T - P,$$

whence

$$T = P \left( 1 + \frac{a}{g} \right).$$

If the lift starts descending with the same acceleration, the tension in the cable will be

$$T = P \left( 1 - \frac{a}{g} \right).$$

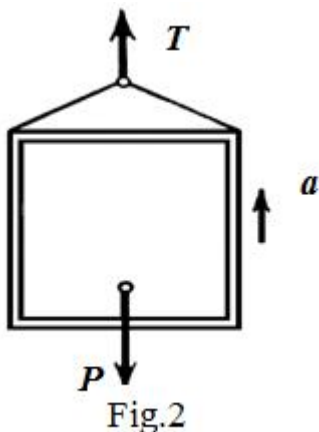


Fig.2

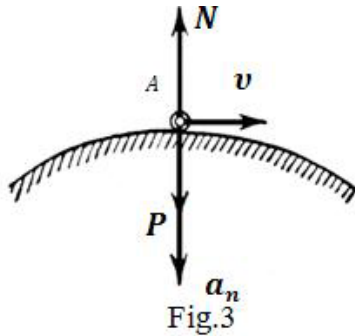
*Problem 3.* The radius of curvature of a bridge at point  $A$  is  $R$  (Fig. 3). Determine the pressure exerted on the bridge at  $A$  by a motor car of weight  $P$  moving with a velocity  $v$ .

*Solution.* The normal acceleration of the car at point  $A$  is

$$a_n = \frac{v^2}{R}.$$

Acting on it are the force of gravity  $P$  and the reaction of constraint  $N$ . Then, from Eq. (1.4),

$$\frac{P v^2}{g R} = P - N,$$



whence

$$N = P \left( 1 - \frac{v^2}{gR} \right).$$

The pressure on the bridge is equal to  $N$  in magnitude but is directed downward.

*Problem 4.* A crank of length  $l$  (Fig. 4) rotates with a uniform angular velocity  $\omega$  and translates the slotted bar  $K$  of weight  $P$  along slides 1,1. Neglecting friction, determine the pressure exerted by the slide block  $A$  on the slotted bar.

*Solution.* The position of the bar is specified by its coordinate  $x = l \cos \omega t$ . Eq. (1.4) for the motion of the bar in terms of its projection on  $x$  axis gives  $ma_x = Q_x$ . But

$$a_x = \frac{d^2 x}{dt^2} = -l\omega^2 \cos \omega t = -\omega^2 x,$$

whence, as  $Q_x = -Q$ ,  $-\frac{P}{g}\omega^2 x = -Q$ ,  $Q = \frac{P}{g}\omega^2 x$ .

Thus, the pressure of the slide block on the slotted bar is proportional to its coordinate  $x$ .

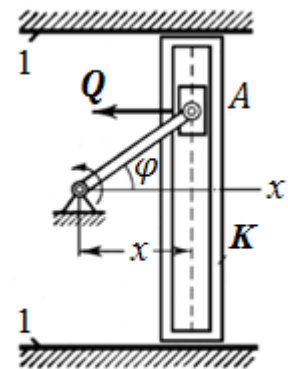
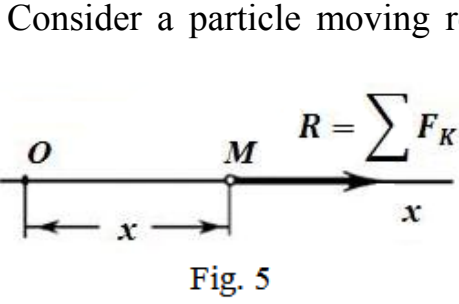


Fig. 4

## 2. DIFFERENTIAL EQUATIONS OF MOTION FOR A PARTICLE AND THEIR INTEGRATION

### 2.1. Rectilinear Motion of a Particle

We know from kinematics that in rectilinear motion the velocity and acceleration of a particle are continuously directed along the same straight line. As the direction of acceleration is coincident with the direction of force, it follows that a free particle will move in a straight line whenever the force acting on it is of constant direction and the velocity at the initial moment is either zero or is collinear with the force.



Consider a particle moving rectilinearly under the action of an applied force  $\mathbf{R} = \sum \mathbf{F}_k$ . The position of the particle on its path is specified by its coordinate  $x$  (Fig. 5). In this case the principal problem of dynamics is: knowing  $\mathbf{R}$ , to find the equation of motion of the particle  $x = f(t)$ . Eq. (1.2) gives the relation between  $x$  and  $\mathbf{R}$ . Projecting both sides of the equation on axis  $Ox$ , we obtain

$$ma_x = R_x = \sum F_{kx} \text{ or as } a_x = \frac{d^2x}{dt^2},$$

$$m \frac{d^2x}{dt^2} = \sum F_{kx}. \quad (2.1)$$

Eq. (2.1) is called the differential equation of rectilinear motion of a particle. It is often more convenient to replace Eq. (2.1) with two differential equations containing first derivatives:

$$m \frac{dv_x}{dt} = \sum F_{kx}, \quad (2.2)$$

$$\frac{dx}{dt} = v_x. \quad (2.2')$$

Whenever the solution of a problem requires that the velocity be found as a function of the coordinate  $x$  instead of time  $t$  (or when the forces themselves depend on  $x$ ), Eq. (2.2') is converted to the variables  $x$ . As  $\frac{dv_x}{dt} = \frac{dv_x}{dx} \times \frac{dx}{dt} = \frac{dv_x}{dx} v_x$  Eq. (2.2) takes the form

$$mv_x \frac{dv_x}{dx} = \sum F_{kx}. \quad (2.3)$$

The principal problem of dynamics is, essentially, to develop the equation of motion  $x=f(t)$  for a particle from the above equations, the forces being known. For this it is necessary to integrate the corresponding differential equation. In order to make clearer the nature of the mathematical problem, it should be recalled that the forces in the right side of Eq. (2.1) can depend on time  $t$ , on the position of the particle  $x$ , or on the velocity  $v_x = \frac{dx}{dt}$ . Consequently, in the general case Eq. (2.1) is, mathematically, a differential equation of the second order in the form

$$\frac{d^2x}{dt^2} = \Phi \left( t, x, \frac{dx}{dt} \right). \quad (2.4)$$

The equation can be solved for every specific problem after determining the form of its right-hand member, which depends on the applied forces. When Eq. (2.4) is integrated for a given problem, the general solution will include two constants of integration  $C_1$  and  $C_2$  and the general form of the solution will be

$$x = f(t, C_1, C_2). \quad (2.5)$$

To solve a concrete problem, it is necessary to determine the values of the constants  $C_1$  and  $C_2$ . For this we introduce the so-called initial conditions.

Investigation of any motion begins from some specified instant called the initial time  $t=0$ , usually the moment when the motion under the action of the given forces starts. The position occupied by a particle at the initial time is called its initial displacement, and its velocity at that time is its initial velocity (a particle can have an initial velocity either because at time  $t=0$  it was moving under no force or because up to time  $t=0$  it was subjected to the action of some other forces). To solve the principal problem of dynamics we must know, besides the applied forces, the initial conditions, i.e., the position and velocity of the particle at the initial time.

In the case of rectilinear motion, the initial conditions are specified in the form

$$\text{at } t = 0, \quad x = x_0, \quad v_x = v_0. \quad (2.6)$$

From the initial conditions we can determine the meaning of the constants  $C_1$  and  $C_2$ , and develop finally the equation of motion for the particle in the form

$$x = f(t, x_0, v_0). \quad (2.7)$$

The following simple example will explain the above. Let there be acting on a particle a force  $\mathbf{Q}$  of constant magnitude and direction. Then Eq. (2.2) acquires the form

$$m \frac{dv_x}{dt} = Q_x.$$

As  $Q_x = \text{const.}$ , multiplying both members of the equation by  $dt$  and integrating, we obtain

$$v_x = \frac{Q_x}{m} t + C_1. \quad (2.8)$$

Substituting the value of  $v_x$  into Eq. (2.2'), we have

$$\frac{dx}{dt} = \frac{Q_x}{m} t + C_1.$$

Multiplying through by  $dt$  and integrating once again, we obtain

$$x = \frac{1}{2} \frac{Q_x}{m} t^2 + C_1 t + C_2. \quad (2.9)$$

This is the general solution of Eq. (2.4) for the specific problem in the form given by Eq. (2.5).

Now let us determine the integration constants  $C_1$  and  $C_2$ , assuming for the specific problem the initial conditions given by (2.6). Solutions (2.8) and (2.9) must satisfy any moment of time, including  $t=0$ . Therefore, substituting zero for  $t$  in Eqs. (2.8) and (2.9), we should obtain  $v_0$  and  $x_0$ , instead of  $v_x$  and  $x$ , i.e., we should have  $v_0 = C_1$ ,  $x_0 = C_2$ .

These equations give the values of the constants  $C_1$  and  $C_2$ , which satisfy the initial conditions of a given problem. Substituting these values into Eq. (2.9), we obtain finally the relevant equation of motion in the form expressed by Eq. (2.7):

$$x = \frac{1}{2} \frac{Q_x}{m} t^2 + v_0 t + x_0. \quad (2.10)$$

We see from Eq. (2.10) that a particle subjected to a constant force performs uniformly variable motion. This could have been foreseen; for, if  $\mathbf{Q} = \text{const.}$ ,  $\mathbf{a} = \text{const.}$ , too. An example of this type of motion is the motion of a particle under



the force of gravity, in which case in Eq. (2.10)  $\frac{Q_x}{m} = g$  and axis  $Ox$  is directed vertically down.

## 2.2. Curvilinear Motion of a Particle

Consider a free particle moving under the action of forces  $\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_n$ . Let us draw a fixed set of axes  $Oxyz$  (Fig. 6). Projecting both members of the equation (1.2) on these axes, and taking into account that  $a_x = \frac{d^2x}{dt^2}, a_y = \frac{d^2y}{dt^2}, a_z = \frac{d^2z}{dt^2}$  we obtain the differential equations of curvilinear motion of a body in terms of the projections on rectangular Cartesian axes:

$$m \frac{d^2x}{dt^2} = \sum F_{kx}, \quad m \frac{d^2y}{dt^2} = \sum F_{ky}, \quad m \frac{d^2z}{dt^2} = \sum F_{kz}. \quad (2.11)$$

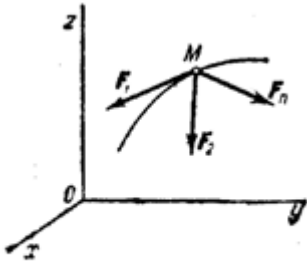


Fig.6

As the forces acting on the particle may depend on time, the displacement or the velocity of the particle, then by analogy with Eq. (2.4), the right-hand members of Eqs. (2.11) may contain the time  $t$ , the coordinates  $x, y, z$  of the particle, and the projections of its velocity  $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ . Furthermore, the right side of each equation may include all these variables.

Eq. (2.11) can be used to solve both the first and the second (the principal) problems of dynamics. To solve the principal problem of dynamics we must know, besides the acting forces, the initial conditions, i.e., the position and velocity of the particle at the initial time. The initial conditions for a set of coordinate axes  $Oxyz$  are specified in the form: at  $t=0$ ,

$$\begin{aligned} x &= x_0, y = y_0, z = z_0, \\ v_x &= v_{x0}, v_y = v_{y0}, v_z = v_{z0}. \end{aligned} \quad (2.12)$$

Knowing the acting forces, by integrating Eq. (2.11), we find the coordinates  $x, y, z$  of the moving particle as functions of time  $t$ , i.e., the equation of motion for the particle. The solutions will contain six constants of integration  $C_1, C_2, \dots, C_6$ , the values of which must be found from the initial conditions (2.12). An example of integrating of Eqs.(2.11) is given in §2.3.

## 2.3. Motion of a Particle Thrown at an Angle to the Horizon in a Uniform Gravitational Field

Let us investigate the motion of a projectile thrown with an initial velocity  $\mathbf{v}_0$  at an angle  $\alpha$  to the horizon, considering it as a material particle of mass  $m$ , neglecting the resistance of the atmosphere, assuming that the horizontal range is small as compared with the radius of the earth and considering the gravitational field to be uniform ( $P = const.$ ).

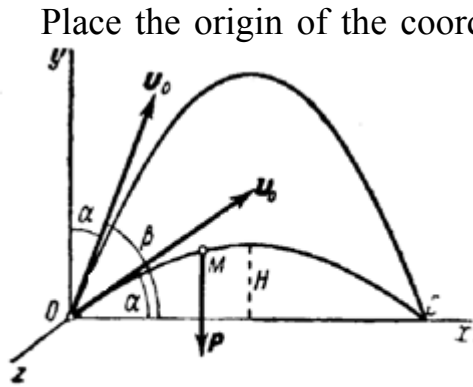


Fig. 7

Place the origin of the coordinate axes  $O$  at the initial position of the particle, direct the  $y$ -axis vertically up, the  $x$ -axis in the plane through  $Oy$  and vector  $\mathbf{v}_0$ , and the  $z$ -axis perpendicular to the first two (Fig. 7). The angle between vector  $\mathbf{v}_0$  and the  $x$ -axis will be  $\alpha$ .

Draw now moving particle  $M$  anywhere on its path. Acting on the particle is only the force of gravity  $\mathbf{P}$ , the projections of which on the coordinate axes are  $P_x = 0$ ,  $P_y = -P = -mg$ ,  $P_z = 0$ .

Substituting these values into Eq.(2.11) and

noting that  $\frac{d^2x}{dt^2} = \frac{dv_x}{dt}$ , etc., after eliminating  $m$ , we obtain:

$$\frac{dv_x}{dt} = 0, \quad \frac{dv_y}{dt} = -g, \quad \frac{dv_z}{dt} = 0.$$

Multiplying these equations by  $dt$  and integrating, we find  $v_x = C_1$ ,  $v_y = -gt + C_2$ ,  $v_z = C_3$ .

The initial conditions of our problem have the form

$$\text{at } x = 0, y = 0, z = 0;$$

$$v_x = v_0 \cos \alpha, \quad v_y = v_0 \sin \alpha, \quad v_z = 0.$$

Satisfying the initial conditions, we have

$$C_1 = v_0 \cos \alpha, \quad C_2 = v_0 \sin \alpha = 0, \quad C_3 = 0.$$

Substituting these values of  $C_1$ ,  $C_2$  and  $C_3$  in the solutions above and replacing  $v_x, v_y, v_z$  by  $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ , we arrive at the equations

$$\frac{dx}{dt} = v_0 \cos \alpha, \quad \frac{dy}{dt} = v_0 \sin \alpha - gt, \quad \frac{dz}{dt} = 0.$$

Integrating, we obtain  $x = v_0 t \cos \alpha + C_4$ ,  $y = v_0 t \sin \alpha - \frac{gt^2}{2} + C_5$ ,  $z = C_6$ .

Substituting the initial conditions, we have  $C_4 = C_5 = C_6 = 0$ . And finally we obtain the equations of motion of particle  $M$  in the form

$$x = v_0 t \cos \alpha, \quad y = v_0 t \sin \alpha - \frac{gt^2}{2}, \quad z = 0. \quad (2.13)$$

From the last equation it follows that the motion takes place in the plane  $Oxy$ .

Knowing the equations of motion of a particle it is possible to determine all the characteristics of the given motion by the methods of Kinematics.

1. *Path.* Eliminating the time  $t$  between the first two of Eqs. (2.13), we obtain the equation of the path of the particle:

$$y = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}. \quad (2.14)$$

This is an equation of a parabola the axis of which is parallel to the  $y$ -axis. Thus, a heavy particle thrown at an angle to the horizon in vacuum follows a parabolic path.

2. *Horizontal Range.* The horizontal range is the distance  $OC=X$  along the  $x$ -axis. Assuming in Eq. (2.14)  $y=0$ , we obtain the points of intersection of the path with the  $x$ -axis. From the equation

$$x \left( \tan \alpha - \frac{gx}{2v_0^2 \cos^2 \alpha} \right) = 0$$

we obtain

$$x_1 = 0, \quad x_2 = \frac{2v_0^2 \cos^2 \alpha \tan \alpha}{g}.$$

The first solution gives point  $O$ , the second point  $C$ . Consequently  $X = x_2$  and finally

$$X = \frac{v_0^2}{g} \sin 2\alpha. \quad (2.15)$$

From Eq. (2.15) we see that the horizontal range  $X$  is the same for angle  $\beta$ , where  $2\beta=180^\circ-2\alpha$ , i.e., if  $\beta=90^\circ-\alpha$ . Consequently, a particle thrown with a given initial velocity  $v_0$  can reach the same point  $C$  by two paths: flat (low) ( $\alpha<45^\circ$ ) or curved (high) ( $\beta=90^\circ-\alpha>45^\circ$ ). With a given initial velocity  $v_0$  the maximum horizontal range in vacuum is obtained when  $\sin 2\alpha = 1$ , i.e., when angle  $\alpha = 45^\circ$ .

3. *Height of path.* If in Eq. (2.14) we assume  $x = \frac{1}{2}X = \frac{v_0^2}{g} \sin \alpha \cos \alpha$ , we obtain the height  $H$  of the path:

$$H = \frac{v_0^2}{2g} \sin^2 \alpha. \quad (2.16)$$

4. *Time of flight.* It follows from Eq. (2.13) that the total time of flight is defined by the equation  $X = v_0 T \cos \alpha$ . Substituting the expression for  $X$ , we obtain

$$T = \frac{2v_0}{g} \sin \alpha. \quad (2.17)$$

At the maximum range angle  $\alpha^*=45^\circ$ , all the quantities become respectively

$$X^* = \frac{v_0^2}{g}, \quad T^* = \frac{\sqrt{2}v_0}{g}, \quad H^* = \frac{v_0^2}{4g} X.$$

## 2.4. Solution of Problems

*Problem 5.* A load of weight  $P$  starts moving from rest along a smooth horizontal plane under the action of a force  $R$  the magnitude of which increases in proportion to the time, the relation being  $R=kt$ . Develop the equation of motion for the load.

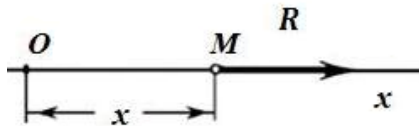


Fig. 8

*Solution.* Place the origin  $O$  in the initial position of the load and direct the axis  $Ox$  in the direction of motion (see Fig. 8). Then the initial conditions are: at  $t=0$ ,  $x=0$  and  $v_x = 0$ . Draw the load in an arbitrary position and the forces acting on it. We have  $R_x = R = kt$ , and Eq.

(2.2) takes the form

$$\frac{P}{g} \frac{dv}{dt} = kt.$$

Multiplying through by  $dt$ , we immediately separate the variables and obtain

$$v_x = \frac{kg}{P} \frac{t^2}{2} + C_1.$$

Substituting the initial values into this equation, we find that  $C_1 = 0$ . Then, substituting  $\frac{dx}{dt}$  for  $v_x$ , we have

$$\frac{dx}{dt} = \frac{kg}{2P} t^2.$$

Multiplying through by  $dt$  we again separate the variables and, integrating, we find

$$x = \frac{kg}{2P} \frac{t^3}{3} + C_2.$$

Substitution of the initial values gives  $C_2 = 0$ , and we obtain the equation of motion for the load in the form

$$x = \frac{kg}{6P} t^3.$$

*Problem 6.* Neglecting the resistance of the air, determine the time it would take a body to travel from end to end of a tunnel  $AB$  dug through the earth along a chord (Fig. 9). Assume the earth's radius to be  $R = 6,370 \text{ km}$ .

Note. The theory of gravitation states that a body inside the earth is attracted towards the centre of the earth with a force  $F$  directly proportional to the distance  $r$  from the centre. Taking into account that, at  $r = R$  (i.e., at the surface of the earth), force  $F$  is equal to the weight of the body ( $F = mg$ ), we find that inside the earth  $F = \frac{mg}{R} r$ , where  $r = MC$  is the distance of point  $M$  from the centre of the earth.

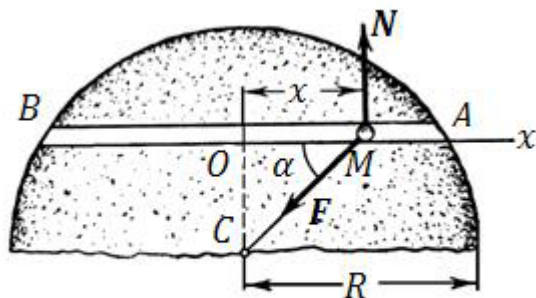


Fig. 9

*Solution.* Place the origin  $O$  in the middle of the chord  $AB$  (where a body in the tunnel would be in equilibrium) and direct the axis  $Ox$  along  $OA$ . If we assume the chord to be of length  $2a$ , initial conditions will be: at  $t = 0$ ,  $x = a$  and  $v_x = 0$ .

The forces acting on the body in an arbitrary position are  $F$  and  $N$ . Consequently,

$$\sum F_{kx} = -F \cos \alpha = -\frac{mg}{R} r \cos \alpha = -\frac{mg}{R} x,$$

as it is evident from the diagram that  $r \cos \alpha = x$ .

We see that the acting force depends on the coordinate  $x$  of point  $M$ . In order to separate the variables in the differential equation of motion, write it in the form (2.3). Then, eliminating  $m$  and introducing the quantity

$$\frac{g}{R} = k^2,$$

we obtain

$$v_x \frac{dv_x}{dx} = -k^2 x.$$

Multiplying through by  $dx$ , we separate the variables and, integrating, obtain

$$\frac{v_x^2}{2} = k^2 \frac{x^2}{2} + C_1.$$

From the initial condition, at  $x = a$ ,  $v_x = 0$ ; hence  $C_1 = \frac{1}{2}k^2 a^2$ . Substituting this expression of  $C_1$ , we have

$$v_x = \pm k\sqrt{a^2 - x^2}.$$

As in the investigated position the velocity is directed from  $M$  to  $O$ ,  $v_x < 0$ , and the sign before the radical should be minus. Then, substituting  $\frac{dx}{dt}$  for  $v_x$ , we have

$$\frac{dx}{dt} = -k\sqrt{a^2 - x^2}.$$

Separating the variables, we write the equation in the form

$$kdt = -\frac{dx}{\sqrt{a^2 - x^2}},$$

and integrating, we obtain

$$kt = \arccos \frac{x}{a} + C_2.$$

Substituting the initial data (at  $t=0$ ,  $x=a$ ) in this equation, we find that  $C_2 = 0$ . The equation of motion for the body in the tunnel is

$$x = a \cos kt.$$

Thus, the body is in harmonic motion with an amplitude  $a$ .

Now let us determine the time  $t_1$  when the body will reach the end  $B$  of the tunnel. At  $B$  the coordinate  $x = -a$ . Substituting this value in the equation of motion, we obtain  $\cos kt_1 = -1$ , whence  $kt_1 = \pi$  and  $t_1 = \frac{\pi}{k}$ . But we have assumed  $k = \sqrt{\frac{g}{R}}$ . Calculating, we find that the time of the motion through the tunnel, given the conditions of the problem, does not depend on the length of the tunnel and is always equal to

$$t_1 = \pi \sqrt{\frac{R}{g}} \approx 42 \text{ min } 11 \text{ sec}.$$

Let us also find the maximum velocity of the body. From the expression for  $v_x$  we see that  $v = v_{max}$  at  $x = 0$ , i.e., at the origin  $O$ . The magnitude of the velocity is

$$v_{max} = ka = a \sqrt{\frac{g}{R}}.$$

If, for example,  $2a = 0.1R = 637 \text{ km}$ , then  $v_{max} \approx 395 \text{ m/sec} = 1,422 \text{ km/h}$ .

**Problem 7.** A boat of weight  $P=400 \text{ N}$  is pushed and receives an initial velocity

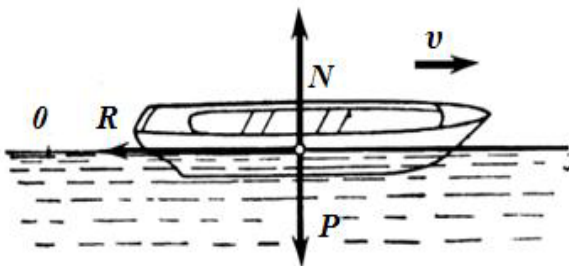


Fig. 10

$v_0 = 0.5 \text{ m/sec}$ . Assuming the resistance of the water at low velocities to be proportional to the first power of the velocity and changing according to the equation  $R = \mu v$ , where the factor  $\mu = 9,3 \text{ N}\cdot\text{sec/m}$ , determine the time in which the velocity will drop by one-half and the distance the boat will travel in that time. Determine also the distance the

boat will travel till it stops.

*Solution.* Let us choose the origin  $O$  to coincide with the initial position of the boat, pointing the axis  $Ox$  in the direction of the motion (Fig.10). In this case the initial conditions will be: at  $t=0$ ,  $x=0$  and  $v_x = v_0$ . Draw the boat in an arbitrary position with the acting forces  $\mathbf{P}$ ,  $\mathbf{N}$ , and  $\mathbf{R}$ .

Calculating the projections of the acting forces, we find that

$$\sum F_{kx} = -R = -\mu v.$$

To determine the duration of the motion, we write differential equation. Noting that  $v_x = v$ , we have

$$\frac{P}{g} \frac{dv}{dt} = -\mu v.$$

Separating the variables, we obtain

$$\frac{dv}{v} = -\frac{\mu g}{P} dt,$$

whence, integrating, we have

$$\ln v = -\frac{\mu g}{P} t + C_1.$$

Substituting the initial values, we have  $C_1 = \ln v_0$ , and finally

$$t = \frac{P}{\mu g} \ln \frac{v_0}{v}.$$

The required time  $t_1$ , is determined by assuming  $v = 0.5v_0$ . We see that in this case the time does not depend on the value of  $v_0$ . As  $\ln 2 = 0.69$ ,

$$t_1 = \frac{P}{\mu g} \ln 2 \approx 3 \text{ sec.}$$

To determine the distance, it is best to write the differential equation of motion in the form (2.3), as it immediately establishes the relation between  $x$  and  $v$ . We thus obtain

$$\frac{P}{g} v \frac{dv}{dx} = -\mu v,$$

whence, eliminating  $v$  and separating the variables, we find

$$dv = -\frac{\mu g}{P} dx,$$

and consequently

$$v = -\frac{\mu g}{P} x + C_1.$$

Since at  $x = 0$  the velocity  $v = v_0$ , then  $C_1 = v_0$ , and finally

$$x = \frac{P}{\mu g} (v_0 - v).$$

Assuming  $v = 0.5v_0$ , we find the required displacement:  $x_1 = \frac{Pv_0}{2\mu g} \approx 1.1 \text{ m.}$

To find the distance travelled by the boat till it stops, in the last equation we assume  $v = 0$ . Then  $x_2 = \frac{Pv_0}{\mu g} = 2.2 \text{ m.}$

### 3. VIBRATION OF A PARTICLE

#### 3.1. Free Harmonic Motion

The study of vibrations is essential for a number of physical and engineering fields. Although the vibrations studied in such different fields as mechanics, radio engineering, and acoustics are of different physical nature, the fundamental laws hold good for all of them. The study of mechanical vibrations is therefore of importance not only because they are frequently encountered in engineering but also because the results obtained in investigating mechanical vibrations can be used in studying and understanding vibration phenomena in other fields.

We shall start with examining free harmonic motion of a particle. Consider a particle  $M$  (Fig.11) moving rectilinearly under the action of a *restoring force*  $F$  directed towards a fixed centre  $O$  and proportional to the distance from that centre. The projection of  $F$  on the axis  $Ox$  is

$$F_x = -cx. \quad (3.1)$$

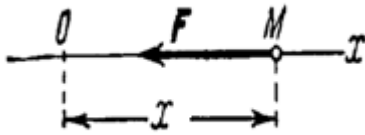


Fig. 11

We see that the force  $F$  tends to return the particle to its position of equilibrium  $O$ , where  $F = 0$ , which is why it is called a "restoring" force. Let us derive the equation of motion of the particle  $M$ . Writing the differential equation of motion (2.1), we

obtain

$$m \frac{d^2x}{dt^2} = -cx.$$

Dividing both sides of the equation by  $m$  and introducing notation

$$\frac{c}{m} = k^2, \quad (3.2)$$

we reduce the equation to the form

$$m \frac{d^2x}{dt^2} + k^2x = 0. \quad (3.3)$$

Eq. (3.3) is the *differential equation of free harmonic motion*. Referring to the theory of differential equations, as the roots of a characteristic equation of the type of Eq. (3.3) are imaginary, its general solution will be

$$x = C_1 \sin kt + C_2 \cos kt, \quad (3.4)$$

where  $C_1$  and  $C_2$  are constants of integration.

If we replace  $C_1$  and  $C_2$  by constants  $a$  and  $\alpha$ , such that  $C_1 = a \cos \alpha$  and  $C_2 = a \sin \alpha$ , we obtain

$$\begin{aligned} x &= a(\sin kt \cos \alpha + \cos kt \sin \alpha) \text{ or} \\ x &= a \sin(kt + \alpha). \end{aligned} \quad (3.5)$$

This is another form of the solution of Eq. (3.3) in which the constants of integration appear as  $a$  and  $\alpha$ , and which is more convenient for general analyses.

The velocity of a particle in this type of motion is

$$v_x = \frac{dx}{dt} = ak \sin(kt + \alpha). \quad (3.6)$$

The vibration of a particle described by Eq. (3.5) is called *simple harmonic motion*.

The quantity  $a$ , which is the maximum distance of  $M$  from the centre of vibration, is called the *amplitude of vibration* (Fig. 12). The quantity  $\varphi = kt + \alpha$  is called the *phase of vibration*. Unlike the coordinate  $x$ , the phase  $\varphi$  defines both the position of the particle at any given time and the direction of its subsequent motion.

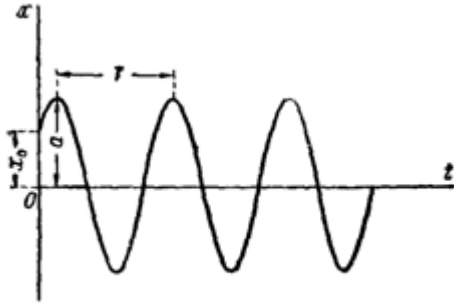


Fig. 12

The quantity  $k$  is called the *angular, or circular, frequency of vibration*. The time  $T$  in which the moving particle makes one complete oscillation is called the *period of vibration*. In one period the phase changes by  $2\pi$ . Consequently, we must have  $kT=2\pi$ , whence the period

$$T = \frac{2\pi}{k}. \quad (3.7)$$

The quantity  $\nu$ , which is the inverse of the period and specifies the number of oscillations per second, is called the *frequency of vibration*:

$$\nu = \frac{1}{T} = \frac{k}{2\pi}.$$

It can be seen from this that the quantity  $k$  differs from  $\nu$  only by a constant multiplier  $2\pi$ . Usually we shall speak of the quantity  $k$  as of frequency.

The values of  $a$  and  $\alpha$  are determined from the initial conditions. Assuming that, at  $t=0$ ,  $x=x_0$  and  $v_x=v_0$  we obtain from Eqs. (3.5) and (3.6)  $x_0 = a \sin \alpha$  and  $\frac{v_0}{k} = a \cos \alpha$ . By first squaring and adding these equations and then dividing them, we obtain

$$a = \sqrt{\frac{v_0^2}{k^2} + x_0^2}, \quad \tan \alpha = \frac{kx_0}{v_0}.$$

Note the following properties of free harmonic motion:

- 1) The amplitude and initial phase *depend* on the initial conditions;
- 2) The frequency  $k$ , and consequently the period  $T$ , *do not depend* on the initial conditions and are invariable characteristics for a given vibrating system.

It follows, in particular, that if a problem requires that only the period (or frequency) of vibration be determined, it is necessary to write a differential equation of motion in the form (3.3). Then  $T$  is found immediately from Eq. (3.7) without integrating.

Consider the next example: a weight is attached to end  $B$  of a vertical spring  $AB$  and released from rest (Fig.13). Determine the law of motion of the weight if the elongation of the spring in the equilibrium condition is  $\delta_{st}$  (the static elongation of the spring).



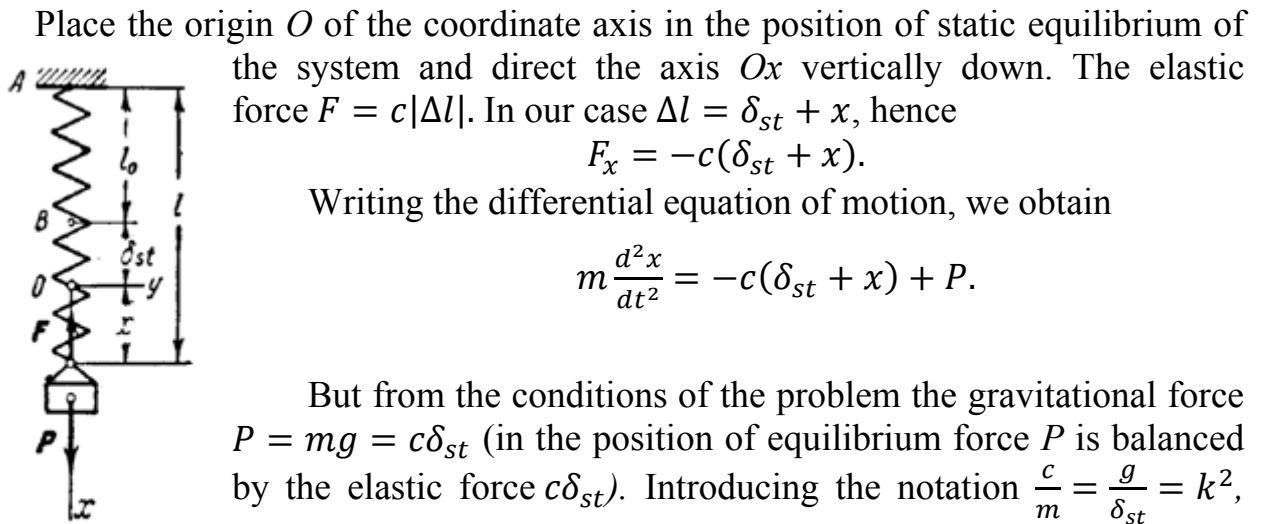


Fig. 13

Place the origin  $O$  of the coordinate axis in the position of static equilibrium of the system and direct the axis  $Ox$  vertically down. The elastic force  $F = c|\Delta l|$ . In our case  $\Delta l = \delta_{st} + x$ , hence

$$F_x = -c(\delta_{st} + x).$$

Writing the differential equation of motion, we obtain

$$m \frac{d^2x}{dt^2} = -c(\delta_{st} + x) + P.$$

But from the conditions of the problem the gravitational force  $P = mg = c\delta_{st}$  (in the position of equilibrium force  $P$  is balanced by the elastic force  $c\delta_{st}$ ). Introducing the notation  $\frac{c}{m} = \frac{g}{\delta_{st}} = k^2$ , we reduce the equation to the form

$$\frac{d^2x}{dt^2} + k^2x = 0,$$

whence immediately we find the period of vibration

$$T = \frac{2\pi}{k} = 2\pi \sqrt{\frac{\delta_{st}}{g}}.$$

Thus, the *period of vibration is proportional to the square root of the static elongation of the spring* (this holds good also for a load vibrating on an elastic beam, where  $\delta_{st}$  is the static deflection of the beam).

The solution of the obtained differential equation is

$$x = C_1 \sin kt + C_2 \cos kt.$$

From the initial conditions, at  $t=0$ ,  $x=\delta_{st}$ , and  $v_x = 0$ . As  $v_x = \frac{dx}{dt} = kC_1 \cos kt - kC_2 \sin kt$ , substituting the initial conditions, we obtain  $C_2 = -\delta_{st}$ ,  $C_1 = 0$ . Hence, the amplitude of vibration is  $\delta_{st}$  and the motion is according to the law

$$x = -\delta_{st} \cos kt.$$

We see that the maximum elongation of the spring in this motion is  $2\delta_{st}$ .

This solution shows that *a constant force  $P$  does not change the type of motion under the action of an elastic force  $F$  but only shifts the center of the vibrations in the direction of the action of the force by the quantity  $\delta_{st}$*  (without the force  $P$  the vibration would, evidently, be about  $B$ ).

### 3.2. Damped Vibration

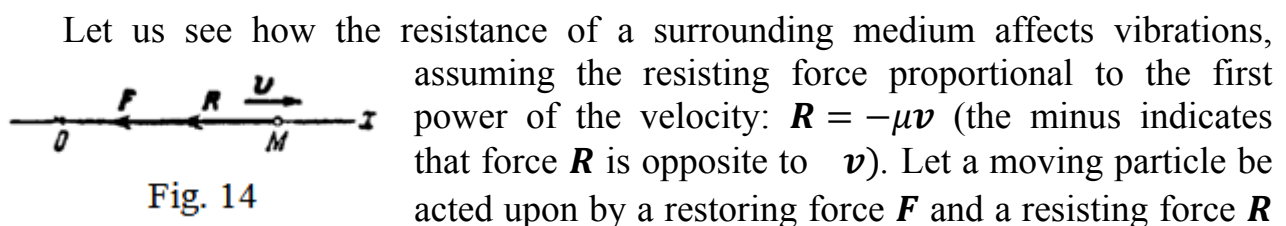


Fig. 14

Let us see how the resistance of a surrounding medium affects vibrations, assuming the resisting force proportional to the first power of the velocity:  $R = -\mu v$  (the minus indicates that force  $R$  is opposite to  $v$ ). Let a moving particle be acted upon by a restoring force  $F$  and a resisting force  $R$

(Fig. 14).

Then  $F_x = -cx$ ,  $R_x = -\mu v_x = -\mu \frac{dx}{dt}$  and the differential equation of motion is

$$m \frac{d^2x}{dt^2} = -cx - \mu \frac{dx}{dt}.$$

Dividing both sides by  $m$ , we obtain

$$\frac{d^2x}{dt^2} + 2b \frac{dx}{dt} + k^2x = 0, \quad (3.8)$$

where

$$\frac{c}{m} = k^2, \quad \frac{\mu}{m} = 2b. \quad (3.9)$$

It is easy to verify that  $k$  and  $b$  have the same dimension ( $\text{sec}^{-1}$ ), which makes it possible to compare them.

Eq. (3.8) is called the *differential equation of damped vibration*. The solution of Eq (3.8) can be found by passing to a new variable  $z$  through the equality  $x = ze^{-bt}$ . Then

$$\frac{dx}{dt} = e^{-bt} \left( \frac{dz}{dt} - bz \right); \quad \frac{d^2x}{dt^2} = e^{-bt} \left( \frac{d^2z}{dt^2} - 2b \frac{dz}{dt} + b^2z \right).$$

Substituting these expressions and the expression of  $x$  into Eq. (3.8), and after the necessary computation, we obtain

$$\frac{d^2z}{dt^2} + (k^2 - b^2)z = 0. \quad (3.10)$$

Let us consider the case when  $k > b$ , i.e., when the resistance is small as compared with the restoring force. Introducing the notation

$$\tilde{k} = \sqrt{k^2 - b^2}, \quad (3.11)$$

we see that Eq. (3.10) coincides with Eq. (3.3). Consequently,  $z = a \sin(\tilde{k}t + \alpha)$  or, passing to  $x$ ,

$$x = ae^{-bt} \sin(\tilde{k}t + \alpha). \quad (3.12)$$

The expression (3.12) gives the solution of differential equation (3.8). The quantities  $a$  and  $\alpha$  are constants of integration and are determined by the initial conditions.

Vibrations according to the law (3.12) are called *damped* because, due to the multiplier  $e^{-bt}$ , the value of  $x$  decreases with time and tends to zero. A graph of such

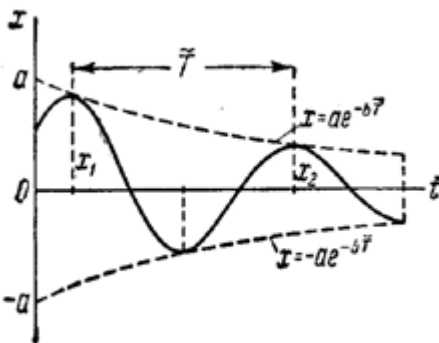


Fig. 15

vibrations is given in Fig. 15. The graph shows that the vibrations are not periodic, though they do show a certain repetition. For example, a particle oscillating about a centre  $O$  returns to that centre at certain intervals  $\tilde{T}$  equal to the period of  $\sin(\tilde{k}t + \alpha)$ .

Therefore, the quantity

$$\tilde{T} = \frac{2\pi}{\tilde{k}} = \frac{2\pi}{\sqrt{k^2 - b^2}} \quad (3.13)$$

is conventionally called the *period of damped vibration*. Comparing, Eqs. (3.13) and (3.7), we see

that  $\tilde{T} > T$ , i.e., that resistance to vibration tends to increase the period of the

vibration. When however, the resistance is small ( $b \ll k$ ) the quantity  $b^2$  can be neglected in comparison with  $k^2$  and we can assume  $\tilde{T} \approx T$ . Thus, a small resistance has no practical effect on the period of vibration.

The time interval between two successive displacements of an oscillating particle to the right or to the left is also equal to  $\tilde{T}$ . Hence, if the maximum displacement  $x_1$  to the right takes place at time  $t_1$  the second displacement  $x_2$  will be at time  $t_2 = t_1 + \tilde{T}$ , etc. Then, by Eq. (3.12) and taking into account that  $\tilde{k}\tilde{T} = 2\pi$ , we have

$$\begin{aligned} x_1 &= ae^{-bt_1} \sin(\tilde{k}t_1 + \alpha), \\ x_2 &= ae^{-b(t_1 + \tilde{T})} \sin(\tilde{k}t_1 + \tilde{k}\tilde{T} + \alpha) = x_1 e^{-b\tilde{T}}. \end{aligned}$$

Similarly, for any displacement  $x_{n+1}$  we will have  $x_{n+1} = x_n e^{-b\tilde{T}}$ . Thus we find that the amplitude of vibration decreases in geometric progression. The denominator of this progression  $e^{-b\tilde{T}}$  is called the *damping decrement*, and the modulus of its logarithm, i.e., the quantity  $b\tilde{T}$ , the *logarithmic decrement*.

It follows from these results that a small resistance has practically no effect on the period of vibration, but gradually damps it by virtue of the amplitude of vibration decreasing according to a law of geometric progression.

When the resistance is large and  $b > k$ , the solution of Eq. (3.10) contains no trigonometric functions. The particle no longer oscillates but instead, under the influence of the restoring force, gradually approaches the position of equilibrium.

### 3.3. Damped Forced Vibrations. Resonance

Consider the motion of a particle on which are acting a restoring force  $F$ , a damping force  $R$  proportional to the velocity (see § 3.2), and a disturbing force  $Q$ , whose projection on the axis  $Ox$  is  $Q_x = Q_0 \sin pt$ . The differential equation of this motion has the form

$$m \frac{d^2x}{dt^2} = -cx - \mu \frac{dx}{dt} + Q_0 \sin pt.$$

Dividing both sides of the equation by  $m$ , assuming  $\frac{Q_0}{m} = P_0$  and taking into account the expression (3.9), we obtain

$$\frac{d^2x}{dt^2} + 2b \frac{dx}{dt} + k^2x = Q_0 \sin pt. \quad (3.14)$$

Eq. (3.14) is the *differential equation of damped forced vibration of a particle*. Its general solution, as is known, has the form  $x = x_1 + x_2$ , where  $x_1$  is the general solution of the equation without the right side, i.e., of Eq. (3.8) [at  $k > b$  this solution is given by Eq. (3.12)], and  $x_2$  is a particular solution of the complete equation (3.14). Let us find the solution  $x_2$  in the form

$$x_2 = A \sin(pt - \beta),$$

where  $A$  and  $\beta$  are constants so chosen that Eq. (3.14) should become an identity. Differentiating, we obtain

$$\frac{dx_2}{dt} = Ap \cos(pt - \beta), \quad \frac{d^2x_2}{dt^2} = -Ap^2 \sin(pt - \beta).$$

Substituting these expressions of the derivatives and  $x_2$  into the left side of Eq. (3.14) and introducing for the sake of brevity the notation  $pt - \beta = \psi$  (or  $pt = \psi + \beta$ ), we obtain

$$A(-p^2 + k^2) \sin \psi + 2bpA \cos \psi = P_0(\cos \beta \sin \psi + \sin \beta \cos \psi).$$

For this equation to be satisfied at any value of  $\psi$ , i.e., at any instant of time, the factors of  $\sin \psi$  and  $\cos \psi$  in the left and right sides should be separately equal. Hence,

$$A(k^2 - p^2) = P_0 \cos \beta, \quad 2bpA = P_0 \sin \beta.$$

First squaring and adding these equations, and then dividing one by the other, we obtain:

$$A = \frac{P_0}{\sqrt{(k^2 - p^2)^2 + 4b^2p^2}}, \quad \tan \beta = \frac{2bp}{k^2 - p^2}. \quad (3.15)$$

As  $x = x_1 + x_2$ , and the expression  $x_1$  is given by Eq. (3.12), we have the final solution of Eq. (3.14) in the form

$$x = ae^{-bt} \sin(\tilde{k}t + \alpha) + A \sin(pt - \beta). \quad (3.16)$$

Here  $a$  and  $\alpha$  are constants of integration determined from the initial conditions, and the expressions for  $A$  and  $\beta$  are given by Eqs. (3.15) and do not depend on the initial conditions. These vibrations are compounded of *natural vibration* [the first term in Eq. (3.16); Fig. 16 a] and *forced vibration* [the second term in Eq. (3.16); Fig. 16 b]. The natural vibration of the particle in such a case was discussed in § 3.2. It was established that it is transient and is damped fairly quickly, and after a certain interval of time  $t_t$  called the *transient period*, can be neglected. A curve showing the transient vibration is given in Fig. 16 c. For practical purposes it can thus be assumed that after a certain transient period a particle will vibrate according to the law

$$x = A \sin(pt - \beta).$$

This is *steady-state forced vibration*, a sustained periodic motion with amplitude  $A$  defined by Eq. (3.15) and a frequency  $p$  equal to the impressed frequency. The quantity  $\beta$  characterizes the *phase shift* of forced vibration with respect to the disturbing force. Let us investigate the results obtained. First let us introduce the notation

$$\frac{p}{k} = \lambda, \quad \frac{b}{k} = h, \quad \frac{P_0}{k^2} = \frac{Q_0}{c} = \delta_0, \quad (3.18)$$

where  $\lambda$  is the frequency ratio,  $h$  a quantity characterizing the damping effect,  $\delta_0$  the magnitude of the static deflection of a particle under the action of force  $Q_0$ .

Then, dividing the numerator and denominator of Eq. (3.15) by  $k^2$ , we obtain

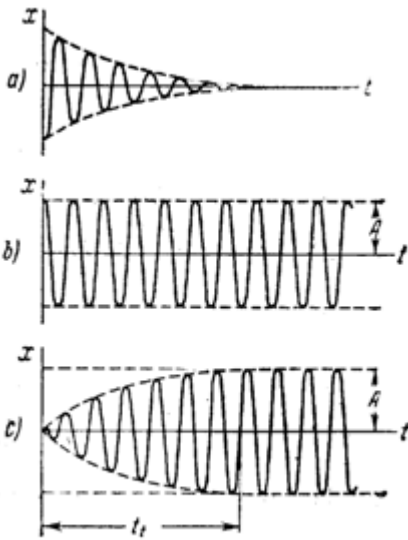


Fig. 16

$$A = \frac{\delta_0}{\sqrt{(1-\lambda^2)^2 + 4h^2\lambda^2}}, \quad \tan \beta = \frac{2h\lambda}{1-\lambda^2}. \quad (3.19)$$

It can be seen from Eq. (3.19) that  $A$  and  $\beta$  depend on two dimensionless parameters  $\lambda$  and  $h$ . Graphs of this relation for certain values of  $h$  are given in Fig. 17. The values of  $\delta_0$ ,  $\lambda$  and  $h$  can be computed for each specific problem from its conditions, and the values of  $A$  and  $p$  determined from the respective graphs or Eqs. (3.19).

These graphs (and equations) also show that by altering the frequency ratio  $\lambda$  we can induce forced vibrations of different amplitude.

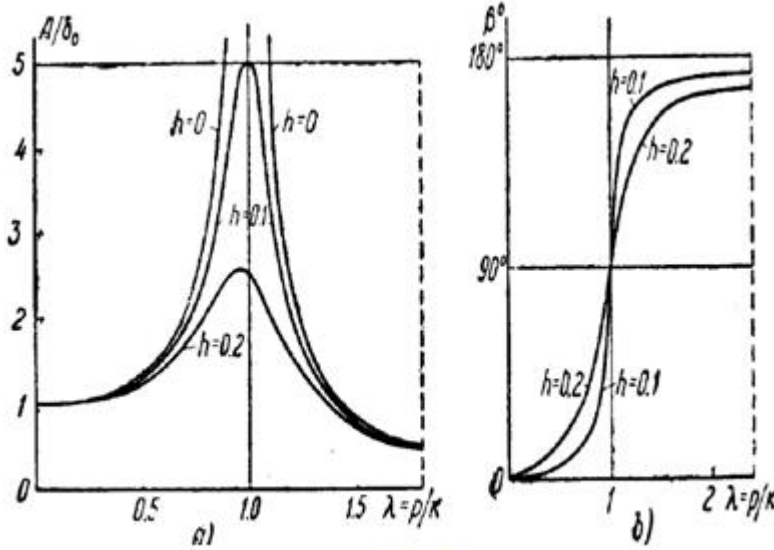


Fig. 17

When the resistance is very small (as ordinarily in the atmosphere) and  $\lambda$  is not close to unity, it is possible in Eqs. (3.19) assume approximately  $h \approx 0$ . In this case we obtain

$$A \approx \frac{\delta_0}{|1-\lambda^2|}; \quad \beta \approx \begin{cases} 0 & (\text{at } \lambda < 1), \\ 180^\circ & (\text{at } \lambda > 1). \end{cases}$$

Let us consider also the following special cases: 1) If the frequency ratio  $\lambda$  is very small ( $p \ll k$ ), then, assuming as an approximation  $\lambda \approx 0$ , we obtain from Eq. (3.19)  $A \approx \delta_0$ . The vibration in this case has an amplitude equal to the static deflection  $\delta_0$  and the phase shift is  $\beta = 0$ .

2) If the frequency ratio  $\lambda$  is very large ( $p \gg k$ ),  $A$  becomes very small. This case is of special interest for the absorption of vibrations in structures, instruments, etc. Assuming the resistance to be small and neglecting  $2h\lambda$  and  $1$  as compared with  $\lambda^2$  in Eq. (3.19), we obtain for computing  $A$  an approximate formula:

$$A = \frac{\delta_0}{\lambda^2} = \frac{P_0}{p^2}.$$

3) In all cases of practical interest  $h$  is very small. Then, from Eq. (3.19), if  $\lambda$  is almost unity the amplitude of forced vibrations becomes very large. This phenomenon is called *resonance*.

At resonance we can assume  $\lambda = 1$  in Eq. (3.19), and then

$$A_r = \frac{\delta_0}{2h}, \quad \beta_r = \frac{\pi}{2}. \quad (3.20)$$

We see that when  $h$  is small  $A_r$  can become very large. When the damping force, and with it  $h$ , tends to zero, the limiting value of the amplitude  $A_r$  as Eq. (3.20) shows, tends to infinity. Thus, with no damping force the vibration amplification process in resonance conditions is unlimited and the amplitude increases indefinitely.

A graph of resonance vibration is given in Fig. 18. When the damping forces are very small the picture is similar.

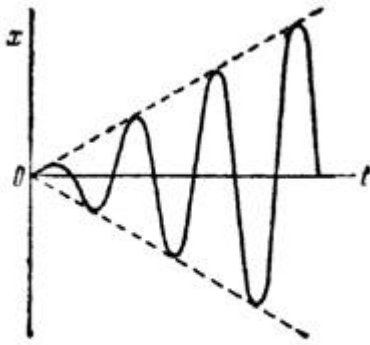


Fig. 18

*General Properties of Forced Vibration.* It follows from the results obtained above that forced vibration has the following important properties, which distinguish it from the natural vibration of a particle:

1) The amplitude of forced vibration does not depend on the initial conditions.

2) Forced vibration does not die out in the presence of resistance.

3) The frequency of forced vibration is equal to the frequency of the disturbing force and does not depend on the characteristics of the vibrating system (the disturbing force "impresses" its own vibration frequency on the system).

4) Even when the disturbing force  $Q$  is small, large forced vibration can be induced if the resistance is small and the frequency  $p$  is almost equal to  $k$  (resonance).

5) Even if the disturbing force is large, forced vibration can be damped if the frequency  $p$  is much larger than  $k$ .

Forced vibration, and resonance in particular, plays an important part in many branches of physics and engineering. Lack of balance in working machines and motors, for example, usually causes forced vibration to appear in the machine or its foundation.

In radio engineering the reverse is true. Resonance is extremely useful and is used to separate the signals of one radio station from those of all others (tuning).

### 3.4. Solution of Problems

*Problem 8.* Determine the periods of vibration of a load of weight  $P$  attached to two springs of stiffness  $C_1$  and  $C_2$  as shown in Figs. 19 and 20.

*Solution.* a) In the first case, in the static position both springs are subjected to a tensile force  $P$ . Therefore, the static elongations are

$$\delta_{1st} = \frac{P}{C_1}, \delta_{2st} = \frac{P}{C_2},$$

and the total elongation is

$$\delta_{st} = \delta_{1st} + \delta_{2st} = \frac{P(C_1 + C_2)}{C_1 C_2}, \text{ and}$$

$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2},$$

where  $C_{eq}$  is the equivalent spring constant of the two given springs. In particular, at  $C_1 = C_2$  we have

$$C_{eq} = \frac{1}{2} C.$$

The period of vibration is

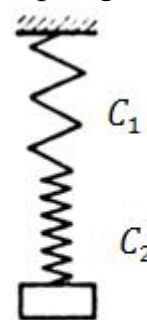


Fig. 19

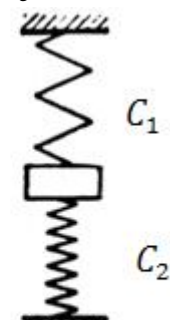


Fig. 20

$$T = 2\pi \sqrt{\frac{\delta_{st}}{g}} = 2\pi \sqrt{\frac{P(C_1+C_2)}{gC_1C_2}}.$$

b) In the second case the top spring is subjected to a tensile force  $P_1$ , and the bottom spring is subjected to a compressive force  $P_2$ , such that  $P_1 + P_2 = P$ . For these springs we have  $\delta_{1st} = \frac{P_1}{c_1}$ ,  $\delta_{2st} = \frac{P_2}{c_2}$ . But obviously  $\delta_{1st} = \delta_{2st} = \delta_{st}$ , and by virtue of the property of proportions

$$\delta_{st} = \frac{P_1}{c_1} = \frac{P_2}{c_2} = \frac{P_1+P_2}{c_1+c_2} \text{ or } \delta_{st} = \frac{P}{c_1+c_2}.$$

The equivalent stiffness  $c_{eq} = c_1 + c_2$  and the period of vibration

$$T = 2\pi \sqrt{\frac{\delta_{st}}{g}} = 2\pi \sqrt{\frac{P}{g(c_1+c_2)}}.$$

*Problem 9.* The deflection caused in a beam by the weight of a motor mounted as shown in Fig. 21 is  $\delta_{st} = 1 \text{ cm}$ . At how many rpm of the shaft will resonance appear?

*Solution.* The period of natural vibration of the beam is

$$T = 2\pi \sqrt{\frac{\delta_{st}}{g}}.$$

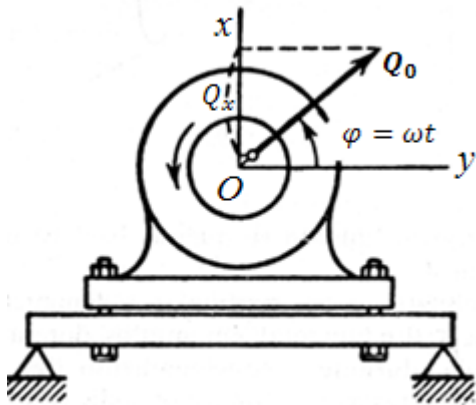


Fig. 21

If the centre of gravity of the shaft is not concentric with its axis, a centrifugal force  $Q_0$  will develop (Fig. 21). Its component  $Q_x = Q_0 \sin \omega t$  (where  $\omega$  is the angular velocity of the shaft) is the disturbing force acting on the beam; its frequency is  $p = \omega$ . Hence, the period of the forced vibration is  $T_f = \frac{2\pi}{\omega}$ .

Resonance will appear when  $T_f = T$ , i.e., at

$$\omega_{cr} = \sqrt{\frac{g}{\delta_{st}}} = 31.3 \text{ sec}^{-1}.$$

Hence, the critical speed

$$n_{cr} = \frac{30\omega_{cr}}{\pi} = 300 \text{ rpm}.$$

The working speed of the motor should be much greater than  $n_{cr}$ .

*Problem 10.* Analyze the forced vibration of a load attached to a spring (see example in p.3.1) if the upper end  $A$  of the spring oscillates vertically according to the law  $\xi = a_0 \sin pt$ .

*Solution.* Draw axis  $Ox$  as in Fig. 13. If we imagine the upper end of the spring displaced from point  $A$  downwards by a quantity  $\xi$ , the length of the spring will be  $l = l_0 - \xi + \delta_{st} + x$ . Then  $F_x = -c\Delta l = -c(\delta_{st} + x - \xi)$ , and the differential equation of motion, neglecting the resistance of the air and taking into account that  $P = c\delta_{st}$ , will be

$$m \frac{d^2x}{dt^2} = -c(\delta_{st} + x - \xi) + P = -cx + c\xi.$$

Introducing the notation  $\frac{c}{m} = k^2$ ,

we obtain

$$\frac{d^2x}{dt^2} + k^2x = k^2a_0 \sin pt.$$

Consequently, the load will experience forced vibration, since, if we assume  $b = 0$  and  $P_0 = k^2a_0$ , the equation coincides with Eq. (3.14). It can be seen from Eq. (3.18) that in the present case  $\delta_0 = a_0$ , and  $h = 0$ . The amplitude of forced vibration and the phase shift are determined by the Eq. (3.19).

If  $p \ll k$  (the top end of the spring oscillates very slowly), then  $\lambda \approx 0$  and  $A \approx a_0$  and the phase shift  $\beta = 0$ . The load will oscillate as if the spring were a rigid rod, which physically corresponds to the condition  $k \gg p$ . At  $p=k$  resonance appears and the amplitude increases sharply. If the frequency  $p$  becomes larger than  $k$  ( $\lambda > 1$ ) the load will vibrate in such a way that it will move down when the end of the spring moves up and vice versa (a phase shift of  $\beta = 180^\circ$ ), and the larger the value of  $p$  the smaller the amplitude. Finally, when  $p$  is much greater than  $k$  ( $\lambda \gg 1$ ), the amplitude  $A \approx 0$ . The load will remain in the position of static equilibrium (point  $O$ ) even though the top end of the spring will oscillate with amplitude  $a_0$  (the frequency of this vibration is so large that the load, as it were, is unable to keep up with it).

## 4. INTRODUCTION TO THE DYNAMICS OF A SYSTEM

### 4.1. Mechanical Systems. External and Internal Forces

A mechanical system is defined as such a collection of material points (particles) or bodies in which the position or motion of each particle or body of the system depends on the position and motion of all the other particles or bodies. We shall thus regard a material body as a system of its particles.

A classical example of a mechanical system is the solar system, all the component bodies of which are connected by the forces of their mutual attraction.

A collection of bodies not connected by interacting forces does not comprise a mechanical system. In this summary we shall consider only mechanical systems, calling them just "systems" for short.

The forces acting on the particles or bodies of a system can be subdivided into external and internal forces.

External forces are defined as the forces exerted on the members of a system by particles or bodies not belonging to the given system. Internal forces are defined as the forces of interaction between the members of the same system. We shall denote external forces by the symbol  $F^e$ , and internal forces by the symbol  $F^i$ . Both external and internal forces can be either active forces or the reactions of constraints. The division of forces into external and internal is purely relative, and it depends on the extent of the system whose motion is being investigated. In considering the motion of



the solar system as a whole, for example, the gravitational attraction of the sun acting on the earth is an internal force; in investigating the earth's motion about the sun, the same force is external.

Internal forces possess the following properties:

1. *The geometrical sum (the principal vector) of all the internal forces of a system is zero.* This follows from the third law of dynamics, which states that any two particles of a system (Fig. 22) act on each other with equal and oppositely directed forces  $\mathbf{F}_{12}^i$  and  $\mathbf{F}_{21}^i$ , the sum of which is zero. Since the same is true for any pair of particles of a system, then

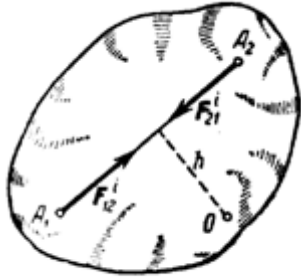


Fig. 22

$$\mathbf{F}_k^i = 0. \quad (4.1)$$

2. *The sum of the moments (the principal moment) of all the internal forces of a system with respect to any centre or axis is zero.* For if we take an arbitrary centre O,

it is apparent from Fig. 22 that  $\mathbf{m}_0(\mathbf{F}_{12}^i) + \mathbf{m}_0(\mathbf{F}_{21}^i) = 0$ . The same result holds good for the moments about any axis. Hence, for the system as a whole we have

$$\sum \mathbf{m}_0(\mathbf{F}_k^i) = 0 \text{ or } \sum m_x(\mathbf{F}_k^i) = 0. \quad (4.2)$$

It does not follow from the above, however, that the internal forces are mutually balanced and do not affect the motion of the system, for they are applied to different particles or bodies and may cause their mutual displacement. The internal forces will be balanced only when a given system is a rigid body.

## 4.2. Mass of a System. Centre of Mass

The motion of a system depends, besides the acting forces, on its total mass and the distribution of this mass. The *mass of a system* is equal to the arithmetical sum of the masses of all the particles or bodies comprising it:

$$M = \sum m_k. \quad (4.3)$$

The distribution of mass is characterized primarily by the location of a point called the centre of mass. *The centre of mass or centre of inertia, of a system is defined as a geometrical point C whose coordinates are given by the equations:*

$$x_c = \frac{\sum m_k x_k}{M}, y_c = \frac{\sum m_k y_k}{M}, z_c = \frac{\sum m_k z_k}{M}, \quad (4.4)$$

where  $m_k$  is the mass of a particle of the system, and  $x_k, y_k, z_k$  are its coordinates.

If the position of a centre of mass is defined by its radius vector  $\mathbf{r}_c$ , we can obtain from Eqs. (4.4) the following expression

$$\mathbf{r}_c = \frac{\sum m_k \mathbf{r}_k}{M}, \quad (4.5)$$

where  $\mathbf{r}_k$  is the radius vector of a particle of the system.

For a body in a uniform gravitational field, the centre of mass coincides with the centre of gravity. The concepts of centre of gravity and centre of mass, however, are not identical. The concept of centre of gravity, as the point through which the resultant of the forces of gravity passes, has meaning only for a rigid body in a uniform field of gravity. The concept of centre of mass, as a characteristic of the

distribution of mass in a system, on the other hand, has meaning for any system of particles or bodies, regardless of whether a given system is subjected to the action of forces or not.

### 4.3. Moment of Inertia of a Body about an Axis. Radius of Gyration

The position of centre of mass does not characterize completely the distribution of mass in a system. For if in the system in Fig. 23 the distance  $h$  of each of two identical spheres  $A$  and  $B$  from the axis  $Oz$  is increased by the same quantity, the

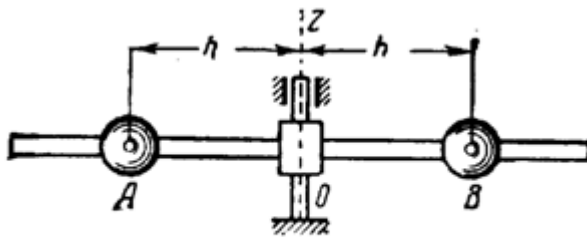


Fig. 23

location of the centre of mass will not change, though the distribution of mass will change and influence the motion of the system (all other conditions remaining the same, the rotation about axis  $Oz$  will be slower).

Accordingly, another characteristic of the distribution of mass, called the moment of inertia, is introduced in mechanics. *The moment of inertia of a body with respect to a given axis  $Oz$  is defined as a scalar quantity equal to the sum of the masses of the particles of the body, each multiplied by the square of its perpendicular distance from the axis*

$$J_z = \sum m_k h_k^2. \quad (4.6)$$

It will be shown further on that moment of inertia plays the same part in the rotational motion of a body as mass does in translational motion, i.e., *moment of inertia is a measure of a body's inertia in rotational motion.*

By Eq. (4.6), the moment of inertia of a body is equal to the sum of the moments of inertia of all its parts with respect to the same axis. For a material point located at a distance  $h$  from an axis,  $J_z = mh^2$ . The dimension of moment of inertia in the international system of units is  $[J] = \text{kg} \cdot \text{m}^2$ .

The concept of *radius of gyration* is often employed in calculations. The radius of gyration of a body with respect to an axis  $Oz$  is a linear quantity  $\rho$  defined by the equation

$$J_z = M\rho^2, \quad (4.7)$$

where  $M$  is the mass of the body.

It follows from the definition that geometrically the radius of gyration is equal to the distance from the axis  $Oz$  to a point, such that if the mass of the whole body were concentrated in it the moment of inertia of the point would be equal to the moment of inertia of the whole body. Knowing the radius of gyration, we can obtain the moment of inertia of a body from Eq. (4.7) and vice versa.

### 4.4. Moments of Inertia of Some Homogeneous Bodies

If we divide a body into elements, in the limit the sum in Eq. (4.6) will become an integral and we obtain

$$J_z = \iiint h^2 dm, \quad (4.8)$$

where the integration is over the whole volume of the body and depends on the coordinates of the points of the body. Eq. (4.8) is convenient in computing the moments of inertia of homogeneous bodies. Let us examine some examples.

1. *Thin Homogeneous Rod of Length  $l$  and Mass  $M$ .* Let us find its moment of inertia with respect to an axis  $Az$  perpendicular to the rod (Fig. 24). If we lay off an axis  $Ax$  along  $AB$ , for any line element of length  $dx$  we have  $h = x$  and its mass  $dm = \rho_1 dx$ , where  $\rho_1 = M/l$  is the mass of a unit length of the rod, and Eq. (4.8) gives:

$$J_A = \int_0^l x^2 dm = \rho_1 \int_0^l x^2 dx = \rho_1 \frac{l^3}{3}.$$

Substituting the expression for  $\rho_1$ , we obtain finally

$$J_A = \frac{1}{3} Ml^3.$$

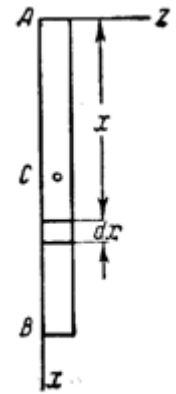


Fig. 24

2. *Thin Circular Homogeneous Ring of Radius  $R$  and Mass  $M$ .* Let us find its moment of inertia with respect to an axis  $Cz$  perpendicular to the plane of the ring through its centre (Fig. 25). As all the points of the ring are at a distance  $h_k = R$  from axis  $Cz$ , Eq. (4.6) gives

$$J_c = \sum m_k R^2 = (\sum m_k) R^2 = MR^2.$$

Hence, for the ring

$$J_c = MR^2.$$

It is evident that the same result is obtained for the moment of inertia of a cylindrical shell of mass  $M$  and radius  $R$  with respect to its axis.

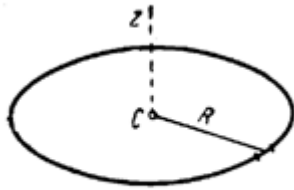


Fig. 25

3. *Circular Homogeneous Disc or Cylinder of Radius  $R$  and Mass  $M$ .* Let us compute the moment of inertia of a circular disc with respect to an axis  $Cz$  perpendicular to it through its centre (Fig. 26a). Consider an elemental ring of radius  $r$  and width  $dr$ . Its area is  $2\pi r dr$ , and its mass  $dm = \rho_2 2\pi r dr$ , where  $\rho_2 = \frac{M}{\pi R^2}$  is the mass of a unit area of the disc. From Eq.(4.8) we have for the elemental ring

$$dJ_c = r^2 dm = 2\pi\rho_2 r^3 dr$$

and for the whole disc

$$J_c = 2\pi\rho_2 \int_0^R r^3 dr = \frac{1}{2} \pi\rho_2 R^4.$$

Substituting the expression for  $\rho_2$  we obtain finally

$$J_c = \frac{1}{2} MR^2.$$

It is evident that the same formula is obtained for the moment of inertia  $J_z$  of a homogeneous circular cylinder of mass  $M$  and radius  $R$  with respect to its axis  $Cz$  (Fig. 26b).

The moments of inertia of non-homogeneous and composite bodies can be determined experimentally with the help of appropriate instruments.

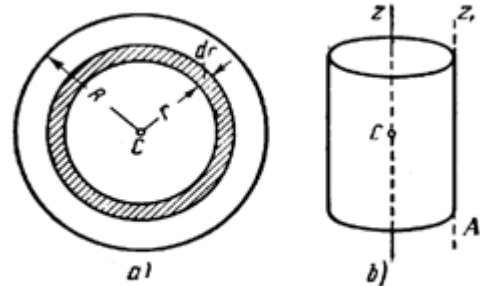


Fig. 26

#### 4.5. Moments of Inertia of a Body about Parallel Axes. The Parallel-Axis (Huygens') Theorem

In the most general case, the moments of inertia of the same body with respect to different axes are different. Let us see how to determine the moment of inertia of a body with respect to any axis if its moment of inertia with respect to a parallel axis through the body is known.

Draw an axis  $Cz$  through the centre of mass  $C$  of a body, and an axis  $Oz_1$  parallel to it (Fig.27), denoting the distance between the two axes by the symbol  $d$ . By definition we have

$$J_{Oz_1} = \sum m_k h_k^2, \quad J_{Cz} = \sum m_k h'_k{}^2,$$

where  $h_k$  is the distance of an arbitrary point  $B$  of the body from axis  $Oz_1$ , and  $h'_k$  is the distance of the same point from axis  $Cz$ . It follows from  $\triangle Bae$  that

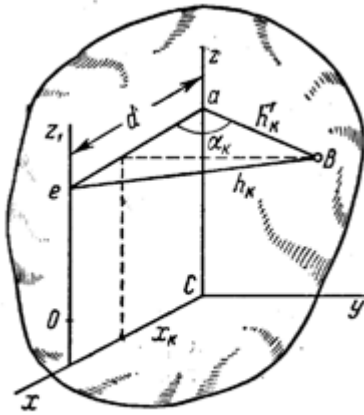


Fig. 27

$$h_k^2 = h'_k{}^2 + d^2 - 2dh'_k \cos \alpha_k.$$

Let us draw from point  $C$ , as the origin of a coordinate system, axes  $x$  and  $y$  perpendicular to  $Cz$ , such that  $x$  intersects with axis  $Oz_1$ . It is evident that  $Cx \parallel ae$ . Denoting the coordinates of point  $B$  as  $x_k, y_k, z_k$ , we obtain

$$h'_k \cos \alpha_k = x_k \text{ and } h_k^2 = h'_k{}^2 + d^2 - 2dx_k.$$

Substituting this expression of  $h_k^2$  into the expression for  $J_{Oz_1}$  and taking the common factors  $d^2$  and  $2d$  outside the summation signs, we have

$$J_{Oz_1} = \sum m_k h'_k{}^2 + (\sum m_k)d^2 - 2d \sum m_k x_k.$$

The first summation in the right side of the equation is equal to  $J_{Cz}$  and the second to the mass  $M$  of the body. Let us find the value of the third summation. From Eq. (4.4) we know that, for the coordinates of the centre of mass,  $\sum m_k x_k = Mx_C$ . But since in our case point  $C$  is the origin,  $x_C = 0$ , and consequently  $\sum m_k x_k = 0$ . We finally obtain

$$J_{Oz_1} = J_{Cz} + Md^2. \quad (4.9)$$

Eq. (4.9) expresses the parallel-axis theorem enunciated by Huygens:

*the moment of inertia of a body with respect to any axis is equal to the moment of inertia of the body with respect to a parallel axis through the centre of mass of the body plus the product of the mass of the body and the square of the distance between the two axes.*

It follows from Eq. (4.9) that  $J_{Oz_1} > J_{Cz}$ . Consequently, of all the axes of same direction, the moment of inertia is least with respect to the one through the centre of mass.



## 5. GENERAL THEOREMS OF DYNAMICS

In solving many problems of dynamics it will be found that the so-called general theorems, representing corollaries of the fundamental law of dynamics, are more conveniently applied than the method of integration of differential equations of motion.

The importance of the general theorems is that they establish visual relationships between the principal dynamic characteristics of motion of material bodies, thereby presenting broad possibilities for analyzing the mechanical motion widely employed in practical engineering. Furthermore, the general theorems make it possible to study for practical purposes specific aspects of a given phenomenon without investigating the phenomenon as a whole. Finally, the use of the general theorems makes it unnecessary to carry out for every problem the operations of integration performed once and for all in proving the theorems, which simplifies the solution.

### 5.1. Momentum of a Particle and a System

One of the basic dynamic characteristics of particle motion is momentum (or linear momentum).

*The momentum of a particle is defined as a vector quantity  $m\mathbf{v}$  equal to the product of the mass of the particle and its velocity. The vector  $m\mathbf{v}$  is directed in the same direction as the velocity, i.e., tangent to the path of the particle.*

*The linear momentum, or simply the momentum, of a system is defined as the vector quantity  $\mathbf{Q}$  equal to the geometric sum (the principal vector) of the moments of all the particles of the system (Fig 28):*

$$\mathbf{Q} = \sum m_k \mathbf{v}_k. \quad (5.1)$$

It can be seen from the diagram that, irrespective of the velocities of the

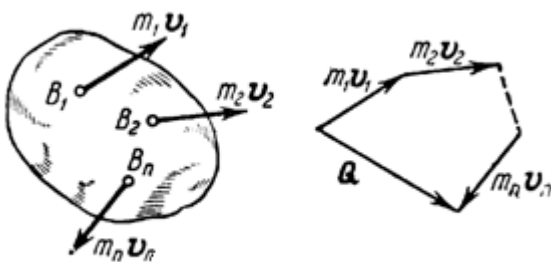


Fig. 28

particles (provided that they are not parallel) the momentum vector can take any value, or even be zero when the polygon constructed with the vectors  $m_k \mathbf{v}_k$  as its sides is closed. Consequently, the quantity  $\mathbf{Q}$  does not characterize the motion of the system completely. Let us develop a formula with which it is much more convenient to

compute  $\mathbf{Q}$  and also to explain its meaning. It follows from Eq. (4.5) that

$$\sum m_k \mathbf{r}_k = M \mathbf{r}_C.$$

Differentiating both sides with respect to time, we obtain

$$\sum m_k \frac{d\mathbf{r}_k}{dt} = M \frac{d\mathbf{r}_C}{dt} \text{ or } \sum m_k \mathbf{v}_k = M \mathbf{v}_C,$$

whence we find that

$$\mathbf{Q} = M\mathbf{v}_C, \quad (5.2)$$

i.e., the momentum of a system is equal to the product of the mass of the whole system and the velocity of its center of mass. This equation is especially convenient in computing the momentum of rigid bodies.

It follows from Eq. (5.2) that, if the motion of a body (or a system) is such that the center of mass remains motionless, the momentum of the body is zero. Thus, the momentum of a body rotating about a fixed axis through its center of mass is zero (the polygon in Fig. 28 is closed).

If, on the other hand, a body has relative motion, the quantity  $\mathbf{Q}$  will not characterize the rotational component of the motion about the center of mass. Thus, for a rolling wheel,  $\mathbf{Q} = M\mathbf{v}_C$ , regardless of how the wheel rotates about its center of mass  $C$ .

We see, therefore, that momentum characterizes only the translational motion of a system, which is why it is often called linear momentum. In relative motion, the quantity  $\mathbf{Q}$  characterizes only the translational component of the motion of a system together with its center of mass.

## 5.2. Impulse of a Force

The concept of impulse (or linear impulse) of a force is used to characterize the effect on a body of a force acting during a certain interval of time. First let us introduce the concept of elementary impulse, i.e., impulse in an infinitesimal time interval  $dt$ . Elementary impulse is defined as a vector quantity  $d\mathbf{S}$  equal to the product of the vector of the force  $\mathbf{F}$  and the time element  $dt$ :

$$d\mathbf{S} = \mathbf{F} dt.$$

The elementary impulse is directed along the action line of the force. The impulse  $\mathbf{S}$  of any force  $\mathbf{F}$  during a finite time interval  $t_1$  is computed as the integral sum of the respective elementary impulses:

$$\mathbf{S} = \int_0^{t_1} \mathbf{F} dt. \quad (5.3)$$

Thus, the impulse of a force in any time interval  $t_1$  is equal to the integral of the elementary impulse over the interval from zero to  $t_1$ .

In the special case when the force  $\mathbf{F}$  is of constant magnitude and direction ( $\mathbf{F} = \text{const.}$ ), we have  $\mathbf{S} = \mathbf{F}t_1$ . In the general case the magnitude of an impulse can be computed from its projections. We can find the projections of an impulse on a set of coordinate axes if we remember that an integral is the limit of a sum, and the projection of a vector sum on an axis is equal to the sum of the projections of the component vectors on the same axis. Hence,

$$S_x = \int_0^{t_1} F_x dt, \quad S_y = \int_0^{t_1} F_y dt, \quad S_z = \int_0^{t_1} F_z dt.$$

With these projections we can construct the vector  $\mathbf{S}$  and find its magnitude and the angles it makes with the coordinate axes. The dimension of impulse in the

international system of units is  $[S] = N \cdot sec$ .

To solve the principal problem of dynamics, it is important to establish the forces whose impulses can be computed without knowing the equation of motion of the particle moving under the action of those forces. It is apparent that to these forces belong only constant forces and forces depending on time.

### 5.3. Theorem of the Motion of Center of Mass

In many cases the nature of the motion of a system (especially of a rigid body) is completely described by the law of motion of its center of mass. To develop this law, let us take the equations of motion of a system (4.10) and add separately their left and right sides. We obtain

$$\sum m_k \mathbf{a}_k = \sum \mathbf{F}_k^e + \sum \mathbf{F}_k^i. \quad (5.4)$$

Let us transform the left side of the equation. For the radius vector of the center of mass we have

$$\sum m_k \mathbf{r}_k = M \mathbf{r}_c.$$

Taking the second derivative of both sides of this equation with respect to time, and noting that the derivative of a sum equals the sum of the derivatives, we find

$$\sum m_k \frac{d^2 \mathbf{r}_k}{dt^2} = M \frac{d^2 \mathbf{r}_k}{dt^2}$$

or

$$\sum m_k \mathbf{a}_k = M \mathbf{a}_c,$$

where  $\mathbf{a}_c$  is the acceleration of the center of mass of the system. As the internal forces of a system give  $\sum \mathbf{F}_k^i = 0$ , by substituting all the developed expressions into Eq. (5.4), we obtain finally:

$$M \mathbf{a}_c = \sum \mathbf{F}_k^e. \quad (5.5)$$

Eq. (5.5) states the theorem of the motion of the center of mass of a system. Its form coincides with that of the equation of motion of a particle of mass  $m=M$  where the acting forces are equal to  $\mathbf{F}_k^e$ . We can therefore formulate the theorem of the motion of the centre of mass as follows: *the center of mass of a system moves as if it were a particle of mass equal to the mass of the whole system to which are applied all the external forces acting on the system.* Projecting both sides of Eq. (5.5) on the coordinate axes, we obtain

$$M \frac{d^2 x_c}{dt^2} = \sum F_{kx}^e, \quad M \frac{d^2 y_c}{dt^2} = \sum F_{ky}^e, \quad M \frac{d^2 z_c}{dt^2} = \sum F_{kz}^e. \quad (5.6)$$

These are the differential equations of motion of the center of mass in terms of the projections on the coordinate axes. The theorem is valuable for the following reasons:

1) It justifies the use of the methods of particle dynamics. It follows from Eqs. (5.6) that the solutions developed on the assumption that a given body is equivalent to a particle define the law of motion of the center of mass of that body. Thus, these solutions have a concrete meaning. In particular, if a body is being translated, its motion is completely specified by the motion of its center of mass, and consequently, a body in translational motion can always be treated as a particle of mass equal to the



mass of the body. In all other cases, a body can be treated as a particle only when the position of its center of mass is sufficient to specify the position of the body.

2) The theorem makes it possible, in developing the equation of motion for the centre of mass of any system, to ignore all unknown internal forces. This is of special practical value.

#### 5.4. The Law of Conservation of Motion of Center of Mass

The following important corollaries arise from the theorem of the motion of center of mass:

1) Let the sum of the external forces acting on a system be zero:

$$\sum \mathbf{F}_k^e = 0.$$

It follows, then, from Eq. (5.5) that  $\mathbf{a}_c = 0$  or  $\mathbf{v}_c = \text{const.}$  Thus, *if the sum of all the external forces acting on a system is zero, the center of mass of that system moves with a velocity of constant magnitude and direction, i.e., uniformly and rectilinearly.* In particular, if the center of mass was initially at rest it will remain at rest. The action of the internal forces, we see, does not affect the motion of the center of mass.

2) Let the sum of the external forces acting on a system be other than zero, but let the sum of their projections on one of the coordinate axes (the  $x$ -axis, for instance), be zero:

$$\sum F_{kx}^e = 0.$$

The first of Eqs. (5.6), then, gives

$$\frac{d^2 x_c}{dt^2} = 0 \text{ or } \frac{dx_c}{dt} = v_{cx} = \text{const.}$$

Thus, *if the sum of the projections on an axis of all the external forces acting on a system is zero, the projection of the velocity of the center of mass of the system on that axis is a constant quantity.* In particular, if at the initial moment  $v_{cx} = 0$ , it will remain zero at any subsequent instant, i.e., the center of mass of the system will not move along the  $x$ -axis ( $x_c = \text{const.}$ ).

The above results express the law of conservation of motion of the center of mass of a system.

#### 5.5. Theorem of the Change in the Momentum of a Particle

As the mass of a particle is constant, and its acceleration  $\mathbf{a} = \frac{d\mathbf{v}}{dt}$  equation, which expresses the fundamental law of dynamics, can be expressed in the form:

$$\frac{d(m\mathbf{v})}{dt} = \sum \mathbf{F}_k. \quad (5.7)$$

Let a particle of mass  $m$  moving under the action of a force  $\mathbf{R} = \sum \mathbf{F}_k$  have a velocity  $\mathbf{v}_0$  at time  $t = 0$ , and at time  $t_1$  let its velocity be  $\mathbf{v}_1$ . Now let us multiply both sides of Eq. (5.7) by  $dt$  and take definite integrals. On the right side, where we integrate with respect to time, the limits of the integrals are zero and  $t_1$ ; on the left side, where we integrate the velocity, the limits of the integral are the respective

values of  $\mathbf{v}_0$  and  $\mathbf{v}_1$ . As the integral of  $d(m\mathbf{v})$  is  $m\mathbf{v}$ , we have

$$m\mathbf{v}_1 - m\mathbf{v}_0 = \sum \int_0^{t_1} \mathbf{F}_k dt.$$

By Eq. (5.3), the integrals on the right side are the impulses of the acting forces. Hence, we finally have

$$m\mathbf{v}_1 - m\mathbf{v}_0 = \sum \mathbf{S}_k. \quad (5.8)$$

Eq. (5.8) states the theorem of the change in the linear momentum of a particle: *the change in the momentum of a particle during any time interval is equal to the geometric sum of the impulses of all the forces acting on the particle during that interval of time.*

In problem solutions, projection equations are often used instead of the vector equation (5.8). Projecting both sides of Eq. (5.8) on a set of coordinate axes, we have

$$\left. \begin{aligned} m v_{1x} - m v_{0x} &= \sum S_{kx} \\ m v_{1y} - m v_{0y} &= \sum S_{ky} \\ m v_{1z} - m v_{0z} &= \sum S_{kz} \end{aligned} \right\} \quad (5.8')$$

In the case of rectilinear motion along the  $x$ -axis, the theorem is stated by the first of these equations.

## 5.6. Theorem of the Change in Linear Momentum of the System

Consider a system of  $n$  particles. Writing the differential equations of motion (4.10) for this system and adding them, we obtain

$$\sum m_k \mathbf{a}_k = \sum \mathbf{F}_k^e + \sum \mathbf{F}_k^i.$$

From the property of internal forces the last summation is zero. Furthermore,

$$\sum m_k \mathbf{a}_k = \frac{d}{dt} (\sum m_k \mathbf{v}_k) = \frac{d\mathbf{Q}}{dt},$$

and we finally have

$$\frac{d\mathbf{Q}}{dt} = \sum \mathbf{F}_k^e. \quad (5.9)$$

Eq. (5.9) states the theorem of the change in the linear momentum of a system in differential form: *the derivative of the linear momentum of a system with respect to time is equal to the geometrical sum of all the external forces acting on the system.* In terms of projections on Cartesian axes we have

$$\frac{dQ_x}{dt} = \sum F_{kx}^e, \quad \frac{dQ_y}{dt} = \sum F_{ky}^e, \quad \frac{dQ_z}{dt} = \sum F_{kz}^e. \quad (5.10)$$

Let us develop another expression for the theorem. Let the momentum of a system be  $\mathbf{Q}_0$  at time  $t = 0$ , and at time  $t_1$  let it be  $\mathbf{Q}_1$ . Multiplying both sides of Eq. (5.9) by  $dt$  and integrating, we obtain

$$\mathbf{Q}_1 - \mathbf{Q}_0 = \sum \int_0^{t_1} \mathbf{F}_k^e dt,$$

or

$$\mathbf{Q}_1 - \mathbf{Q}_0 = \sum \mathbf{S}_k^e, \quad (5.11)$$

as the integrals to the right give the impulses of the external forces. Eq. (5.11) states the theorem of the change in the linear momentum of a system in integral form: *the change in the linear momentum of a system during any time interval is equal to the*

sum of the impulses of the external forces acting on the system during the same interval of time. In terms of projections on Cartesian axes we have

$$\left. \begin{aligned} Q_{1x} - Q_{0x} &= \sum S_{kx}^e \\ Q_{1y} - Q_{0y} &= \sum S_{ky}^e \\ Q_{1z} - Q_{0z} &= \sum S_{kz}^e \end{aligned} \right\} \quad (5.11')$$

Let us show the connection between this theorem and the theorem of the motion of center of mass. As  $\mathbf{Q} = M\mathbf{v}_c$ , by substituting this expression into Eq. (5.9) and taking into account that  $\frac{d\mathbf{v}_c}{dt} = \mathbf{a}_c$  we obtain  $M\mathbf{a}_c = \sum \mathbf{F}_k^e$ , i.e., Eq. (5.5).

Consequently, the theorem of the motion of center of mass and the theorem of the change in the momentum of a system are, in fact, two forms of the same theorem. Whenever the motion of a rigid body (or system of bodies) is being investigated, both theorems may be used, though Eq. (5.5) is usually more convenient.

For a continuous medium (a fluid), however, the concept of center of mass of the whole system is virtually meaningless, and the theorem of the change in the momentum of a system is used in the solution of such problems.

The practical value of the theorem is that it enables us to exclude from consideration the immediately unknown internal forces (for instance, the reciprocal forces acting between the particles of a liquid).

### 5.7. The Law of Conservation of Linear Momentum

The following important corollaries arise from the theorem of the change in the momentum of a system:

- 1) Let the sum of all the external forces acting on a system be zero:

$$\sum \mathbf{F}_k^e = 0.$$

It follows from Eq. (5.9) that in this case  $\mathbf{Q} = \text{const.}$  Thus, *if the sum of all the external forces acting on a system is zero, the momentum vector of the system is constant in magnitude and direction.*

- 2) Let the external forces acting on a system be such that the sum of their projections on any axis  $Ox$  is zero:

$$\sum F_{kx}^e = 0.$$

It follows from Eqs. (5.10) that in this case  $Q_x = \text{const.}$  Thus, *if the sum of the projections on any axis of all the external forces acting on a system is zero, the projection of the momentum of that system on that axis is a constant quantity.*

These results express the law of conservation of the linear momentum of a system.

### 5.8. Theorem of the Change in the Angular Momentum of a Particle

Often, in analyzing the motion of a particle, it is necessary to consider the change not of the vector  $m\mathbf{v}$  itself, but of its moment. The moment of the vector  $m\mathbf{v}$  with respect to any center  $\theta$  or axis  $z$  is denoted by the symbol  $\mathbf{m}_\theta(m\mathbf{v})$  or  $m_z(m\mathbf{v})$

and is called the moment of momentum or angular momentum with respect to that center or axis. The moment of vector  $m\mathbf{v}$  is calculated in the same way as the moment of a force. Vector  $m\mathbf{v}$  is considered to be applied to the moving particle. In magnitude  $|\mathbf{m}_0(m\mathbf{v})| = mvh$ , where  $h$  is the perpendicular distance from  $O$  to the position line of the vector  $m\mathbf{v}$  (see Fig. 29).

1. *Principle of Moments About an Axis.* Consider a particle of mass  $m$  moving under the action of a force  $\mathbf{F}$ . Let us establish the dependence between the moments of the vectors  $m\mathbf{v}$  and  $\mathbf{F}$  with respect to any fixed axis  $z$ .

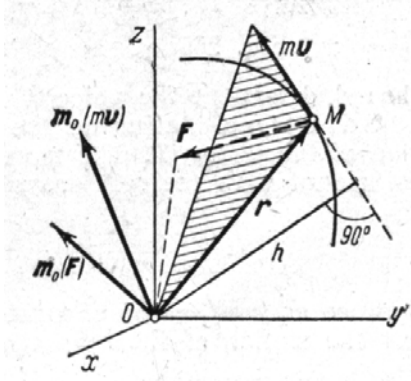


Fig.29

It is well known that

$$m_z(\mathbf{F}) = xF_y - yF_x. \quad (5.12)$$

Similarly, for  $m_z(m\mathbf{v})$ , and taking  $m$  out of the parentheses, we have

$$m_z(m\mathbf{v}) = m(xv_y - yv_x). \quad (5.13)$$

Differentiating both sides of this equation with respect to time, we obtain

$$\frac{d}{dt}[m_z(m\mathbf{v})] = m\left(\frac{dx}{dt}v_y - \frac{dy}{dt}v_x\right) + \left(xm\frac{dv_y}{dt} - ym\frac{dv_x}{dt}\right).$$

The expression in the first parentheses of the right side of the equation is zero, as  $\frac{dx}{dt} = v_x$  and  $\frac{dy}{dt} = v_y$ . From Eq. (5.12), the expression in the second pair of parentheses is equal to  $m_z(\mathbf{F})$ , since, from the fundamental law of dynamics,

$$m\frac{dv_x}{dt} = F_x, m\frac{dv_y}{dt} = F_y.$$

Finally, we have

$$\frac{d}{dt}[m_z(m\mathbf{v})] = m_z(\mathbf{F}). \quad (5.14)$$

This equation states the principle of moments about an axis: *the derivative of the angular momentum of a particle about any axis with respect to time is equal to the moment of the acting force about the same axis*

2. *Principle of Moments about a Center.* Let us find for a particle moving under the action of a force  $\mathbf{F}$  (Fig. 29) the relation between the moments of vectors  $m\mathbf{v}$  and  $\mathbf{F}$  with respect to any fixed center  $O$ . It was shown that  $\mathbf{m}_0(\mathbf{F}) = \mathbf{r} \times \mathbf{F}$ . Similarly,

$$\mathbf{m}_0(m\mathbf{v}) = \mathbf{r} \times m\mathbf{v}.$$

Vector  $\mathbf{m}_0(\mathbf{F})$  is normal to the plane through  $O$  and vector  $\mathbf{F}$ , while the vector  $\mathbf{m}_0(m\mathbf{v})$  is normal to the plane through the center  $O$  and vector  $m\mathbf{v}$ . Differentiating the expression  $\mathbf{m}_0(m\mathbf{v})$  with respect to time, we obtain

$$\frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \left(\frac{d\mathbf{r}}{dt} \times m\mathbf{v}\right) + \left(\mathbf{r} \times m\frac{d\mathbf{v}}{dt}\right) = (\mathbf{v} \times m\mathbf{v}) + (\mathbf{r} \times m\mathbf{a}).$$

But  $\mathbf{v} \times m\mathbf{v} = 0$ , as the vector product of two parallel vectors, and  $m\mathbf{a} = \mathbf{F}$ . Hence,

$$\frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \mathbf{r} \times \mathbf{F}, \text{ or } \frac{d}{dt}[\mathbf{m}_0(m\mathbf{v})] = \mathbf{m}_0(\mathbf{F}). \quad (5.15)$$

This is the principle of moments about a center: *the derivative of the angular momentum of a particle about any fixed center with respect to time is equal to the moment of the force acting on the particle about the same center.* An analogous theorem is true for the moments of vector  $m\mathbf{v}$  and force  $\mathbf{F}$  with respect to any axis  $z$ , which is evident if we project both sides of Eq. (5.15) on that axis. This was proved directly in item 1.

### 5.9. Total Angular Momentum of a System

The total angular momentum of a system with respect to any center  $O$  is defined as the quantity  $\mathbf{K}_O$ , equal to the geometrical sum of the angular momenta of all the particles of the system with respect to that center:

$$\mathbf{K}_O = \sum \mathbf{m}_O(m_k \mathbf{v}_k). \quad (5.16)$$

The angular moment of a system with respect to each of three rectangular coordinate axes are found similarly:

$$K_x = \sum m_x(m_k v_k), K_y = \sum m_y(m_k v_k), K_z = \sum m_z(m_k v_k). \quad (5.17)$$

By the theorem proved in § 5.8,  $K_x$ ,  $K_y$ ,  $K_z$  are the respective projections of vector  $\mathbf{K}_O$  on the coordinate axes.

To understand the physical meaning of  $\mathbf{K}_O$ , let us compute the angular momentum of a rotating body with respect to its axis of rotation. If a body rotates about a fixed axis  $Oz$  (Fig. 30), the linear velocity of any particle of the body at a distance  $h_k$  from the axis is  $\omega h_k$ . Consequently, for that particle

$$m_z(m_k \mathbf{v}_k) = m_k v_k h_k = m_k \omega h_k^2.$$

Then, taking the common multiplier  $\omega$  outside of the parentheses, we obtain for the whole body

$$K_z = \sum m_z(m_k \mathbf{v}_k) = (\sum m_k h_k^2)\omega.$$

The quantity in the parentheses is the moment of inertia of the body with respect to the  $z$ -axis (§ 4.3). We finally obtain

$$K_z = J_z \omega. \quad (5.18)$$

Thus, *the angular momentum of a rotating body with respect to the axis of rotation is equal to the product of the moment of inertia of the body and its angular velocity.*

If a system consists of several bodies rotating about the same axis, then, apparently,

$$K_z = J_{1z}\omega_1 + J_{2z}\omega_2 + \dots + J_{nz}\omega_n.$$

The analogy between Eqs. (5.2) and (5.18) will be readily noticed: the momentum of a body is the product of its mass (the quantity characterizing the body's inertia in translational motion) and its velocity; the angular momentum of a body is equal to the product of its moment of inertia (the quantity characterizing a body's inertia in rotational motion) and its angular velocity.

Just as the momentum of a system is a characteristic of its translational motion,

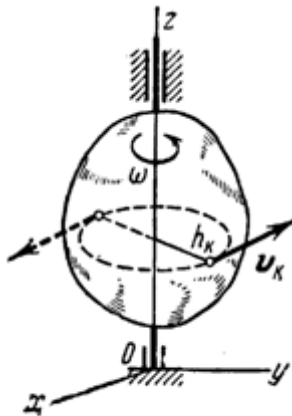


Fig. 30

the total angular momentum of a system is a characteristic of its rotational motion.

### 5.10. Theorem of the Change in the Total Angular Momentum of a System

The principle of moments, which was proved for a single particle (§ 5.8), is valid for all the particles of a system. If, therefore, we consider a particle of mass  $m_k$  and velocity  $\mathbf{v}_k$  belonging to a system, we have for that particle

$$\frac{d}{dt}[\mathbf{m}_0(m_k \mathbf{v}_k)] = \mathbf{m}_0(\mathbf{F}_k^e) + \mathbf{m}_0(\mathbf{F}_k^i).$$

where  $\mathbf{F}_k^e$  and  $\mathbf{F}_k^i$  are the resultants of all the external and internal forces acting on the particle.

Writing such equations for all the particles of the system and adding them, we obtain

$$\frac{d}{dt} \sum [\mathbf{m}_0(m_k \mathbf{v}_k)] = \sum \mathbf{m}_0(\mathbf{F}_k^e) + \sum \mathbf{m}_0(\mathbf{F}_k^i).$$

But from the properties of the internal forces of a system, the last summation vanishes. Hence, taking into account Eq. (5.16), we obtain finally

$$\frac{d\mathbf{K}_0}{dt} = \sum \mathbf{m}_0(\mathbf{F}_k^e). \quad (5.19)$$

This equation states the following principle of moments for a system: *The derivative of the total angular momentum of a system about any fixed center with respect to time is equal to the sum of the moments of all the external forces acting on that system about that center.*

Projecting both sides of Eq. (5.19) on a set of fixed axes  $Oxyz$ , we obtain

$$\frac{dK_x}{dt} = \sum m_x(\mathbf{F}_k^e)_x, \quad \frac{dK_y}{dt} = \sum m_y(\mathbf{F}_k^e)_y, \quad \frac{dK_z}{dt} = \sum m_z(\mathbf{F}_k^e)_z. \quad (5.20)$$

### 5.11. The Law of Conservation of the Total Angular Momentum

The following important corollaries can be derived from the principle of moments.

1) Let the sum of the moments of all the external forces acting on a system with respect to a center  $O$  be zero:

$$\sum \mathbf{m}_0(\mathbf{F}_k^e) = 0.$$

It follows, then, from Eq. (5.19) that  $\mathbf{K}_0 = \text{const.}$  Thus, *if the sum of the moments of all external forces acting on a system taken with respect to any center is zero, the total angular momentum of the system with respect to that center is constant in magnitude and direction.*

2) Let the external forces acting on a system be such that the sum of their moments with respect to any fixed axis  $Oz$  is zero:

$$\sum m_z(\mathbf{F}_k^e)_z = 0.$$

It follows, then, from Eqs. (5.20) that  $K_z = \text{const.}$  Thus, *if the sum of the moments of all the external forces acting on a system with respect to any axis is zero,*

the total angular momentum of the system with respect to that axis is constant.

These conclusions express the law of conservation of the total angular momentum of a system. It follows from them that internal forces cannot change the total angular momentum of a system.

### 5.12. Kinetic Energy of Particle and a System

The kinetic energy of a particle is a scalar quantity equal  $\frac{1}{2}mv^2$ . The kinetic energy of a system is defined as a scalar quantity  $T$  equal to the arithmetical sum of the kinetic energies of all the particles of the system:

$$T = \sum \frac{m_k v_k^2}{2}. \quad (5.21)$$

If a system consists of several bodies, its kinetic energy is, evidently, equal to the sum of the kinetic energies of all the bodies:

$$T = \sum T_k.$$

Let us develop the equations for computing the kinetic energy of a body in different types of motion.

*Translational Motion.* In this case all the points of a body have the same velocity, which is equal to the velocity of the centre of mass. Therefore, for any point  $v_k = v_c$ , and Eq. (5.21) gives

$$\begin{aligned} T_{trans} &= \sum \frac{m_k v_k^2}{2} = \frac{1}{2} (\sum m_k) v_c^2, \\ T_{trans} &= \frac{1}{2} M v_c^2. \end{aligned} \quad (5.22)$$

*Rotational Motion.* The velocity of any point of a body rotating about an axis  $Oz$  is  $v_k = \omega h_k$ , where  $h_k$  is the distance of the point from the axis of rotation, and  $\omega$  is the angular velocity of the body. Substituting this expression into Eq. (5.21) and taking the common multipliers outside the parentheses we obtain

$$T_{rotation} = \sum \frac{m_k \omega^2 h_k^2}{2} = \frac{1}{2} (\sum m_k h_k^2) \omega^2.$$

The term in the parentheses is the moment of inertia of the body with respect to the axis  $z$ . Thus we finally obtain

$$T_{rotation} = \frac{1}{2} J_z \omega^2. \quad (5.23)$$

*Plane Motion.* In plane motion, the velocities of all the points of a body are at any instant directed as if the body were rotating about an axis perpendicular to the plane of motion and passing through the instantaneous centre of zero velocity  $P$  (Fig. 31). Hence, by Eq. (5.23)

$$T_{plane} = \frac{1}{2} J_p \omega^2.$$

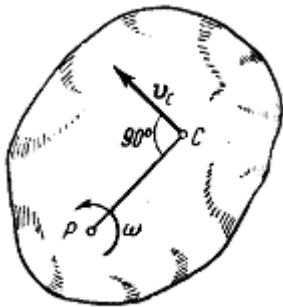


Fig. 31

where  $J_p$  is the moment of inertia of the body with respect to the instantaneous axis of rotation.

The quantity  $J_p$  is variable, as the position of the centre  $P$  continuously changes with the motion of the body. Let us introduce instead of  $J_p$  a constant moment of inertia  $J_c$  with respect to an axis through the centre of mass  $C$  of the body. By the parallel-axis theorem,  $J_p = J_c + Md^2$ , where  $d=PC$ .

Substituting this expression for  $J_p$  and taking into account that point  $P$  is the instantaneous centre of zero velocity and therefore  $\omega d = \omega PC = v_c$ , where  $v_c$  is the velocity of the centre of mass, we obtain finally

$$T_{plane} = \frac{1}{2} M v_c^2 + \frac{1}{2} J_c \omega^2. \quad (5.24)$$

### 5.13. Work Done by a Force. Power

The concept of work is introduced as a measure of the action of a force on a body in a given displacement, specifically that action which is represented by the change in the magnitude of the velocity of a moving particle.

First let us introduce the concept of elementary work done by a force in infinitesimal displacement  $ds$ . The elementary work done by a force  $\mathbf{F}$  (Fig. 32) is defined as a scalar quantity

$$dA = F_\tau ds, \quad (5.25)$$

where  $F_\tau$  is the projection of the force on the tangent to the path in the direction of the displacement, and  $ds$  is an infinitesimal displacement of the particle along that tangent.

This definition corresponds to the concept of work as a characteristic of that action of a force which tends to change the magnitude of velocity. For if force  $\mathbf{F}$  is resolved into components  $\mathbf{F}_\tau$  and  $\mathbf{F}_n$ , only the component  $\mathbf{F}_\tau$ , which imparts the particle its tangential acceleration, will change the magnitude of the velocity. Noting that  $F_\tau = F \cos \alpha$ , we further obtain from Eq. (5.25)

$$dA = F ds \cos \alpha. \quad (5.26)$$

If angle  $\alpha$  is acute, the work is of positive sense. In particular, at  $\alpha = 0$ , the elementary work  $dA = F ds$ .

If angle  $\alpha$  is obtuse, the work is of negative sense. In particular, at  $\alpha = 180^\circ$ , the elementary work  $dA = -F ds$ .

If angle  $\alpha = 90^\circ$ , i.e., if a force is directed perpendicular to the displacement, the elementary work done by the force is zero.

Let us now find an analytical expression for elementary work. For this we resolve force  $F$  into components  $F_x, F_y, F_z$ , parallel to the coordinate axes (Fig. 33). The infinitesimal displacement  $MM'=ds$  is compounded of the displacements  $dx, dy, dz$  parallel to the coordinate axes, where  $x, y, z$  are the coordinates of point  $M$ . The work done by force  $F$  in the displacement  $ds$  can be calculated as the sum of the work

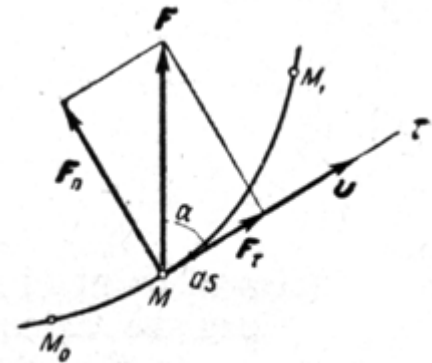


Fig. 32



done by its components  $F_x$ ,  $F_y$ ,  $F_z$  in the displacements  $dx$ ,  $dy$ ,  $dz$ .

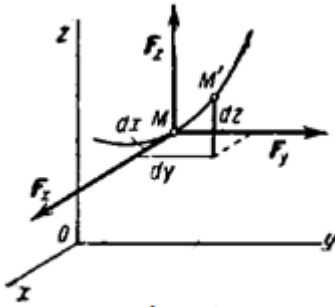


Fig. 33

But the work in the displacement  $dx$  is done only by component  $F_x$  and is equal to  $F_x dx$ . The work in the displacements  $dy$  and  $dz$  is calculated similarly. Thus, we finally obtain

$$dA = F_x dx + F_y dy + F_z dz. \quad (5.27)$$

Eq. (5.27) gives the analytical expression of the elementary work done by a force.

The work done by a force in any finite displacement  $M_0M_1$  (see Fig. 32) is calculated as the integral sum of the corresponding elementary works and is equal to

$$A_{(M_0M_1)} = \int_{(M_0)}^{(M_1)} F_\tau ds, \quad (5.28)$$

$$A_{(M_0M_1)} = \int_{(M_0)}^{(M_1)} (F_x dx + F_y dy + F_z dz). \quad (5.29)$$

The limits of the integral correspond to the values of the variables of integration at points  $M_0$  and  $M_1$ , (or, more exactly, the integral is taken along the curve  $M_1M_0$ , i.e., it is curvilinear).

In order to solve the principal problem of dynamics, it is important to establish the forces whose work can be calculated immediately without knowing the equation of motion of the particle on which they are acting. It can be seen that to these forces belong only constant forces or forces which depend on the position (coordinates) of a moving particle.

*Power.* The term *power* is defined as the work done by a force in a unit of time (the time rate of doing work). If work is done at a constant rate, the power

$$N = \frac{A}{t_1},$$

where  $t_1$  is the time in which the work  $A$  is done. In the general case,

$$N = \frac{dA}{dt} = \frac{F_\tau ds}{dt} = F_\tau v.$$

It can be seen from the equation  $N = F_\tau v$  that if a motor has a given power  $N$ , the tractive force  $F_\tau$  is inversely proportional to the velocity  $v$ . That is why, for instance, on an upgrade or poor road a motor car goes into lower gear, thereby reducing the speed and developing a greater tractive force with the same power.

## 5.14. Examples of Calculation of Work

The examples considered below give results which can be used immediately in solving problems.

*Work Done by a Gravitational Force.* Let a particle  $M$  subjected to a gravitational force  $\mathbf{P}$  be moving from a point  $M_0(x_0, y_0, z_0)$  to a point  $M_1(x_1, y_1, z_1)$ . Choose a coordinate system so that the axis  $Oz$  would point vertically up (Fig. 34). Then  $P_x = 0$ ,  $P_y = 0$ ,  $P_z = -P$ . Substituting these expressions into Eq. (5.29) and taking into account that the integration variable is  $z$ , we obtain

$$A_{(M_0M_1)} = \int_{(M_0)}^{(M_1)} (-P dz) = -P \int_{z_0}^{z_1} dz = P(z_0 - z_1).$$

If point  $M_0$  is higher than  $M_1$  then  $z_0 - z_1 = h$ , where  $h$  is the vertical displacement of the particle; if, on the other hand,  $M_0$  is below  $M_1$  then  $z_0 - z_1 = -(z_1 - z_0) = -h$ . Finally we have

$$A_{(M_0M_1)} = \pm Ph.$$

The work is positive if the initial point is higher than the final one and negative if it is lower.

It follows from this that the work done by gravity does not depend on the path along which the point of its application moves.

Forces possessing this property are called

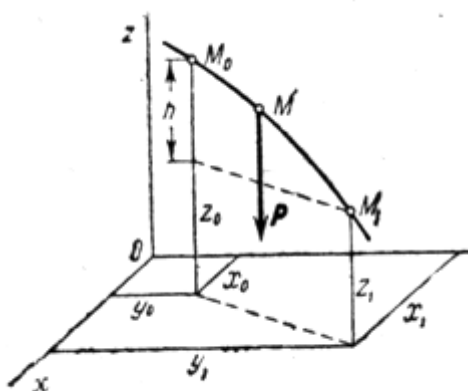


Fig. 34

conservative forces.

*Work Done by an Elastic Force.* Consider a weight  $M$  lying in a horizontal plane and attached to the free end of a spring (Fig. 35). Let point  $O$  on the plane represent the position of the end of the spring when it is not in tension ( $AO = l_0$  is the length of the unextended spring) and let it be the origin of our coordinate system.

Now if we draw the weight from its position of equilibrium  $O$ , stretching the spring to length  $l$ , acting on the weight will be the elastic force of the spring  $F$  directed towards  $O$ . According to Hooke's Law, the magnitude of this force is proportional to the extension of the spring  $\Delta l = l - l_0$ .

As in our case  $\Delta l = x$ , then in magnitude  $F = c|\Delta l| = c|x|$ . The factor  $c$  is called the stiffness of the spring, or the spring constant. Let us find the work done by the elastic force in the displacement of the weight from position  $M_0(x_0)$  to position  $M_1(x_1)$ . As in this case  $F_x = -F = -cx$ ,  $F_y = F_z = 0$ , then, substituting these expressions into Eq. (5.29), we obtain

$$A_{(M_0M_1)} = \int_{(M_0)}^{(M_1)} (-cx) dx = -c \int_{x_0}^{x_1} x dx = \frac{c}{2} (x_0^2 - x_1^2).$$

In the obtained formula  $x_0$  is the initial extension of the spring  $\Delta l_{in}$ , and  $x_1$  is the final extension  $\Delta l_{fin}$ . Hence

$$A_{(M_0M_1)} = \frac{c}{2} [(\Delta l_{in})^2 - (\Delta l_{fin})^2].$$

The work is positive if  $|\Delta l_{in}| > |\Delta l_{fin}|$ , i.e., when the end of the spring moves towards the position of equilibrium, and negative when  $|\Delta l_{in}| < |\Delta l_{fin}|$  i.e., when the end of the spring moves away from the position of equilibrium.

It follows, therefore, that the work done by the force  $F$  depends only on the quantities  $\Delta l_{in}$  and  $\Delta l_{fin}$  and does not depend on the actual path traveled by  $M$ .

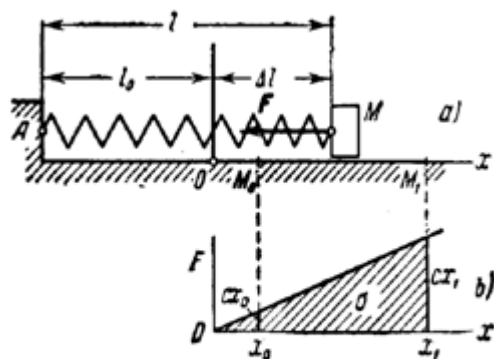


Fig. 35

Consequently, an elastic force is also a conservative force.

*Work Done by Friction.* Consider a particle moving on a rough surface (Fig. 36) or a rough curve. The magnitude of the frictional force acting on the particle is  $fN$ , where  $f$  is the coefficient of friction and  $N$  is the normal reaction of the surface.

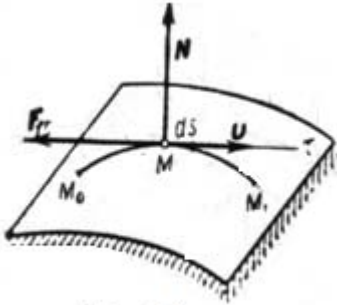


Fig. 36

Frictional force is directed opposite to the displacement of the particle, whence  $F_{fr\tau} = -fN$ , and from Eq. (5.28),

$$A_{(M_0M_1)} = - \int_{(M_0)}^{(M_1)} F_{fr} ds = - \int_{(M_0)}^{(M_1)} fN ds.$$

If the frictional force is constant, then  $A_{(M_0M_1)} = -F_{fr}s$  where  $s$  is the length of the arc  $M_0M_1$  along which the particle moves. Thus, the work done by kinetic friction is always negative. It depends on the length of the arc  $M_0M_1$  and consequently it is non-conservative.

*Work Done by Gravitational Forces Acting on a System.* The work done by a gravitational force acting on a particle of weight  $p_k$  will be  $p_k(z_{k0} - z_{k1})$ , where  $z_{k0}$  and  $z_{k1}$  are the coordinates of the initial and final positions of the particle. Then the total work done by all the gravitational forces acting on a system will be

$$A = \sum p_k z_{k0} - \sum p_k z_{k1} = P(z_{c0} - z_{c1}) = \pm Ph_c.$$

where  $P$  is the weight of the system, and  $h_c$  is the vertical displacement of the centre of gravity (or centre of mass) of the system.

*Work Done by Forces Applied to a Rotating Body.* The elemental work done by the force  $\mathbf{F}$  applied to the body in Fig. 37 will be

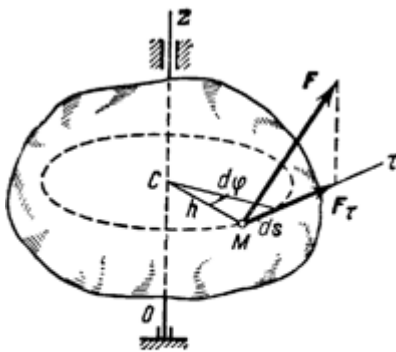


Fig. 37

$$dA = F_\tau ds = F_\tau h d\varphi$$

since  $ds = h d\varphi$ , where  $d\varphi$  is the angle of rotation of the body.

But it is evident that  $F_\tau h = m_z(\mathbf{F})$ . We shall call the quantity  $M_z = m_z(\mathbf{F})$  the *turning moment*, or *torque*. Thus we obtain

$$dA = M_z d\varphi. \quad (5.30)$$

Eq. (5.30) is valid when several forces are acting, if it is assumed that  $M_z = \sum m_z(\mathbf{F}_k)$ . The work done in a turn through a finite angle  $\varphi_1$  will be

$$A = \int_0^{\varphi_1} M_z d\varphi. \quad (5.31)$$

and, for a constant torque ( $M_z = \text{const.}$ ),

$$A = M_z \varphi_1. \quad (5.32)$$

If acting on a body is a force couple laying in a plane normal to  $Oz$ , then, evidently,  $M_z$  in Eqs. (5.30)-(5.32) will denote the moment of that couple.

Let us see how power is determined in this case. From Eq. (5.30) we find

$$N = \frac{dA}{dt} = \frac{M_z d\varphi}{dt} = M_z \omega.$$

*Work Done by Frictional Forces Acting on a Rolling Body.* A wheel of radius  $R$  rolling without slipping on a plane (surface) is subjected to the action of a frictional force  $F_{fr}$ , which prevents the slipping of the point of contact  $B$  on the surface (Fig. 38).

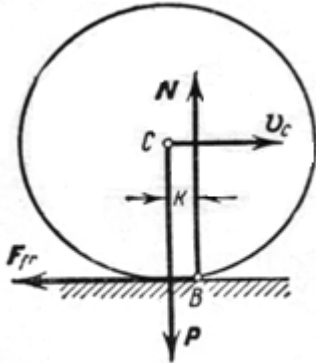


Fig. 38

The elemental work done by this force is  $dA = -F_{fr} ds_B$ . But point  $B$  is the instantaneous centre of zero velocity, and  $v_B = 0$ . As  $ds_B = v_B dt$ ,  $ds_B = 0$ , and for every elemental displacement  $dA = 0$ .

Thus, in rolling without slipping, the work done by the frictional force preventing slipping is zero in any displacement of the body. For the same reason, the work done by the normal reaction  $N$  is also zero. The resistance to rolling is created by the couple  $(N, P)$  of moment  $M = kN$ , where  $k$  is the coefficient of rolling

friction. Then by Eq.(5.30), and taking into account that the angle of rotation of a rolling wheel is  $d\varphi = \frac{ds_C}{R}$ ,

$$dA_{roll} = -kN d\varphi = -\frac{k}{R} N ds_C, \quad (5.33)$$

where  $ds_C$ , is the elemental displacement of the centre  $C$  of the wheel. If  $N = const.$ , then the total work done by the forces resisting rolling will be

$$A_{roll} = -kN\varphi_1 = -\frac{k}{R} N s_C. \quad (5.34)$$

### 5.15. Theorem of the Change in the Kinetic Energy of a Particle

Consider a particle of mass  $m$  displaced by acting forces from a position  $M_0$  where its velocity is  $v_0$ , to position  $M_1$  where its velocity is  $v_1$ .

To obtain the required relation, consider the equation  $m\mathbf{a} = \sum \mathbf{F}_k$ , which expresses the fundamental law of dynamics. Projecting both parts of this equation on the tangent  $M\tau$  to the path of the particle in the direction of motion, we obtain

$$ma_\tau = \sum F_{k\tau}.$$

The tangential acceleration in the left side of the equation can be written in the form

$$a_\tau = \frac{dv}{dt} = \frac{dv}{ds} \frac{ds}{dt} = \frac{dv}{ds} v,$$

hence, we have  $mv \frac{dv}{ds} = \sum F_{k\tau}$ .

Multiplying both sides of the equation by  $ds$ , bring  $mv$  under the differential sign. Then, noting that  $F_{k\tau} ds = dA_k$ , where  $dA_k$  is the elementary work done by the force  $\mathbf{F}_k$ , we obtain an expression of the theorem of the change in kinetic energy in differential form:

$$d\left(\frac{mv^2}{2}\right) = \sum_k dA_k. \quad (5.35)$$

Integrating both sides of Eq. (5.35) in the limits between the corresponding

values of the variables at points  $M_0$  and  $M_1$ , we finally obtain

$$\frac{mv_1^2}{2} - \frac{mv_0^2}{2} = \sum_k A_k. \quad (5.36)$$

Eq. (5.36) states the theorem of the change in the kinetic energy of a particle in the final form: *the change in the kinetic energy of a particle in any displacement is equal to the algebraic sum of the work done by all the forces acting on the particle in the same displacement.*

*The Case of Constrained Motion.* If the motion of a particle is constrained, then the right side of Eq. (5.36) will include the work done by the given (active) forces  $\mathbf{F}_k^a$  and the work done by the reaction of the constraint. Let us limit ourselves to the case of a particle moving on a fixed smooth (frictionless) surface or curve. In this case the reaction  $N$  is normal to the path of the particle, and  $N_\tau = 0$ . Then by Eq. (5.28), the work done by the reaction of a fixed smooth surface (or curve) in any displacement of a particle is zero, and from Eq. (5.36) we obtain

$$\frac{mv_1^2}{2} - \frac{mv_0^2}{2} = \sum A_{(M_0M_1)}^a. \quad (5.36')$$

Thus, in a displacement of a particle on a fixed smooth surface (or curve) the change in the kinetic energy of the particle is equal to the sum of the work done in this displacement by the active forces applied to that particle.

If the surface, (curve) is not smooth, the work done by frictional force will be added to the work done by the active forces.

## 5.16. Theorem of the Change in the Kinetic Energy of a System

The theorem proved in § 5.15 is valid for any point of a system. Therefore, if we take any particle of mass  $m_k$  and velocity  $\mathbf{v}_k$  belonging to a system, we have for this particle

$$\frac{m_k v_{k1}^2}{2} - \frac{m_k v_{k0}^2}{2} = A_k^e + A_k^i,$$

where  $v_{k0}$  and  $v_{k1}$  denote the particle's velocity at the beginning and the end of the displacement, and  $A_k^e$  and  $A_k^i$  are the sums of the work done by all the external and internal forces acting on the particle through this displacement.

If we write similar equations for all the particles of a system and add them up, we obtain

$$\sum \frac{m_k v_{k1}^2}{2} - \sum \frac{m_k v_{k0}^2}{2} = \sum A_k^e + \sum A_k^i \text{ or } T_1 - T_0 = \sum A_k^e + \sum A_k^i, \quad (5.37)$$

where  $T_1$  and  $T_0$  denote the kinetic energy of the system at the beginning and the end of the displacement.

This equation states the following theorem of the change in kinetic energy: *The change in the kinetic energy of a system during any displacement is equal to the sum of the work done by all the external and internal forces acting on the system in that displacement.*

For an infinitesimal displacement of a the system theorem takes the form

$$dT = dA^e - dA^i, \quad (5.38)$$

where  $dA^e$  and  $dA^i$  denote the elemental work done by all the external and internal forces acting on the system. Unlike the previously proved theorems, in Eqs. (5.37) and (5.38) the internal forces are not ignored. For, if  $\mathbf{F}_{12}^i$  and  $\mathbf{F}_{21}^i$  are the forces of interaction between points  $B_1$  and  $B_2$  of a system (see Fig. 39), then  $\mathbf{F}_{12}^i + \mathbf{F}_{21}^i = 0$ , but at the same time point  $B_1$  may be moving towards  $B_2$  and point  $B_2$  towards  $B_1$ . The work done by each force is positive, and the total work will not be zero.

*The Case of Non-Deformable Systems.* A non-deformable system is defined as one in which the distance between the points of application of the internal forces does not change during the motion of the system.

Special cases of such systems are a rigid body and an inextensible string. Let two points  $B_1$  and  $B_2$  of a non-deformable system (Fig.39) be acting on each other with forces  $\mathbf{F}_{12}^i$  and  $\mathbf{F}_{21}^i$  ( $\mathbf{F}_{12}^i = -\mathbf{F}_{21}^i$ ) and let their velocities at some instant be  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . Their displacements in a time interval  $dt$  will be  $ds_1 = v_1 dt$  and

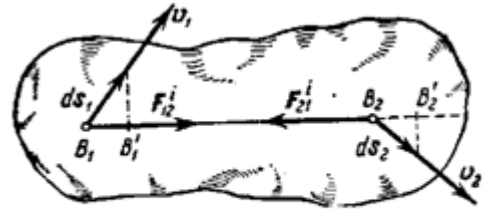


Fig. 39

But as line  $B_1B_2$  is non-deformable, it follows from the laws of kinematics that the projections of vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  and consequently of the displacements  $ds_1$  and  $ds_2$  on the direction of  $B_1B_2$  will be equal, i.e.,  $B_1B_1' = B_2B_2'$ . Then the elemental work done by forces  $\mathbf{F}_{12}^i$  and  $\mathbf{F}_{21}^i$  will be equal in magnitude and opposite in sense, and their sum will be zero. This holds for all internal forces in any displacement of a system.

We conclude from this that the sum of the work done by all the internal forces of a non-deformable system is zero, and Eq. (5.37) takes the form

$$T_1 - T_0 = \sum A_k^e. \quad (5.39)$$

Both the external and internal forces in Eqs. (5.37)-(5.39) include the reactions of constraints. If the constraints on which the bodies of a system move are smooth, then the work done by the reactions of these constraints in any displacement of the system is zero and the reactions will not enter into Eqs (5.37)-(5.39).

Thus in applying the theorem of the change in kinetic energy to frictionless systems, all the immediately unknown reactions of the constraints will be excluded from the problem. This is where its practical value lies.

## 5.17. Solution of Problems

*Problem 11.* The centre of gravity of the shaft of the motor in Fig. 40 is located at a distance  $AB = a$  from the axis of rotation. The shaft is of weight  $\mathbf{p}$ , and the weight of all other parts of the motor is  $\mathbf{P}$ . Deduce the law of motion of the motor on a smooth horizontal surface if the shaft rotates with a uniform angular velocity  $\omega$ . Also determine the maximum stress that will be developed in a bolt  $D$  fastening the motor to the surface.

*Solution.* In order to eliminate the forces rotating the shaft by making them internal, consider the motor with the shaft as a single system.

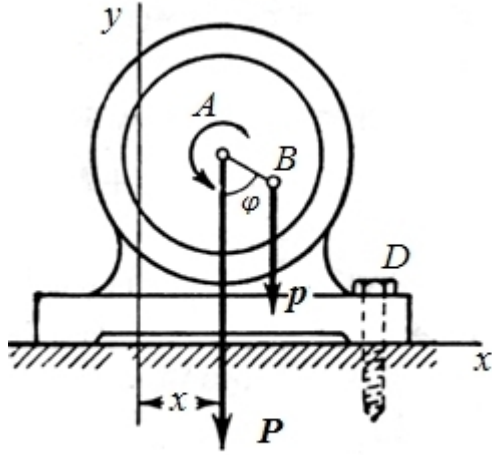


Fig. 40

1) For the motor standing freely on the plane, all the forces acting on it are vertical and the law of conservation of the motion of the centre of mass parallel to axis  $Ox$  will apply. Show the motor in an arbitrary position, assuming as initial the position in which points  $B$  and  $A$  are on the same vertical (on the  $y$  axis). Then in the arbitrary position  $\xi_A = x$ ,  $\xi_B = x + a \sin \varphi$ . Here  $\xi_A$  and  $\xi_B$  perform projections of absolute displacements on  $x$  axis. Hence, taking into account the law of conservation of motion

of the center of mass of a system ( $x_c = \text{const.}$ ) and that  $\varphi = \omega t$ , we find

$$P\xi_A + p\xi_B = 0 \text{ or } Px + p(x + a\sin\omega t) = 0,$$

whence

$$x = -\frac{pa}{P+p} \sin \omega t.$$

Thus, the motor will perform simple harmonic motion with an angular frequency  $\omega$ .

2) When the motor is fastened, the horizontal reaction  $R_x$  of the bolt, by the first of Eqs. (5.6), will be

$$R_x = M \frac{d^2 x_c}{dt^2}, \text{ where } x_c = \frac{Px_A + px_B}{Mg}.$$

In this case point  $A$  is fixed, and  $x_A = h$  ( $h = \text{const.}$ ) and  $x_B = h + a \sin \omega t$ . Differentiating the expression of  $x_c$  and multiplying it by  $M$ , where  $M$  is the mass of the whole system, we obtain

$$R_x = M \frac{d^2 x_c}{dt^2} = \frac{p}{g} \frac{d^2 x_B}{dt^2} = -\frac{pa\omega^2}{g} \sin \omega t.$$

The pressure on the bolt is equal to  $|R_x|$  in magnitude and opposite in direction. Its maximum value will be

$$\frac{pa\omega^2}{g}.$$

*Problem 12.* Crank  $AB$  of length  $r$  and weight  $p$  of the mechanism in Fig. 41 rotates with a constant angular velocity  $\omega$  and actuates the slotted bar and the piston  $D$  connected to it. The total weight of the bar and piston is  $P$ . Acting on the piston during the motion is a constant force  $Q$ . Neglecting friction, determine the maximum horizontal pressure of the crank on its axle  $A$ .

*Solution.* In order to eliminate the forces rotating the crank and the pressure exerted on it by the slotted bar, consider the motion of the system as a whole.

Denoting the horizontal reaction of the axle  $A$  by  $R_x$  we have from the first of Eqs. (5.6)

$$M \frac{d^2 x_C}{dt^2} = R_x - Q,$$

where, by Eqs. (4.4),  $Mx_C = m_1 x_1 + m_2 x_2$ .

In our case

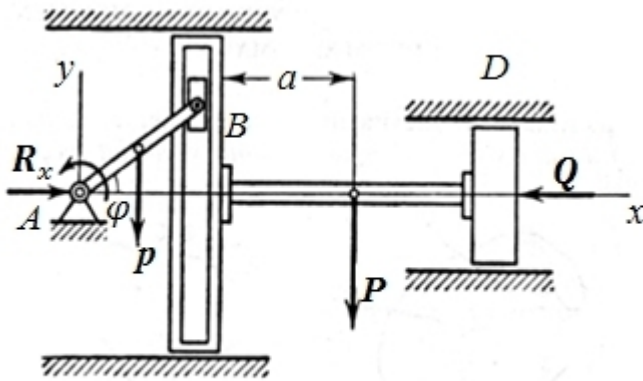


Fig. 41

$$m_1 = \frac{p}{g}, \quad x_1 = \frac{r}{2} \cos \omega t;$$

$$m_2 = \frac{P}{g}, \quad x_2 = a + r \cos \omega t,$$

as  $\varphi = \omega t$ . We finally obtain

$$R_x = Q + M \frac{d^2 x_C}{dt^2} = Q - \frac{r\omega^2}{g} \left( \frac{p}{2} + P \right) \cos \omega t.$$

The pressure on the shaft is equal in magnitude to  $|R_x|$  and oppositely directed. The maximum pressure will

be at  $\varphi = 180^\circ$  and will be equal to

$$Q + \frac{r\omega^2}{g} \left( \frac{p}{2} + P \right).$$

*Problem 13.* A load of weight  $p = 1 \text{ N}$  moves in a circle with a constant velocity  $2 \text{ m/sec}$ . Determine the impulse and the work done by the force acting on the load during the time the load takes to travel one quarter of the circle.

*Solution.* From the theorem of the change in momentum,

$$S = m\mathbf{v}_1 - m\mathbf{v}_0.$$

Constructing geometrically the difference between these momenta (Fig. 42), we find from the right-angled triangle:

$$S = m\sqrt{v_1^2 + v_0^2}.$$

But from the conditions of the problem  $v_1 = v_0 = v$ , consequently

$$S = \frac{p}{g} v\sqrt{2} = 0.29 \text{ N} \cdot \text{sec}.$$

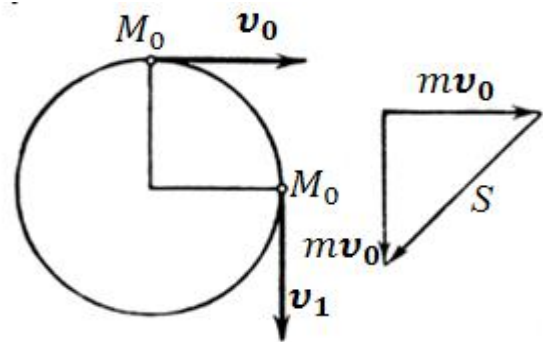


Fig. 42

*Problem 14.* A load of mass  $m$  lying on a horizontal plane is pushed and imparted an initial velocity  $\mathbf{v}_0$ . The motion of the load is then retarded by a constant force  $\mathbf{F}$ . Determine the time it takes the load to stop and how far it will have travelled.

*Solution.* Draw the load in an arbitrary position (Fig. 43),  $M_0$  and  $M_1$  being its initial and final positions. Acting on the load are its weight  $\mathbf{P}$ , the reaction of the plane  $\mathbf{N}$ , and the opposing force  $\mathbf{F}$ . Pointing axis  $Ox$  in the direction of the motion, we have from Eq. (5.8')



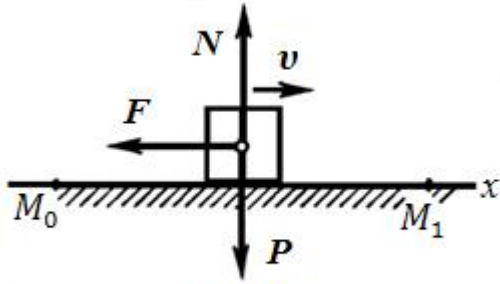


Fig. 43

$$mv_{1x} - mv_{0x} = \sum S_x.$$

In this case  $v_{1x} = 0$  ( $v_1$  is the velocity at the instant when the load stops) and  $v_{0x} = v_0$ . Force  $F$  is the only one projected on the  $x$  axis. As it is constant,  $S_x = F_x t_1 = -F t_1$ , where  $t_1$  is the deceleration time. Substituting these expressions into our equation, we obtain  $-mv_0 = -F t_1$ , whence the required time is

$$t_1 = \frac{mv_0}{F}.$$

To determine the braking distance we use the theorem of the change in kinetic energy:

$$\frac{mv_1^2}{2} - \frac{mv_0^2}{2} = \sum A_k.$$

Here again  $v_1 = 0$  and only force  $F$  does the work:  $A(F) = -Fs$ , where  $s$  is the braking distance. The work done by forces  $P$  and  $N$  is zero as they are perpendicular to the displacement. Hence we obtain  $-\frac{mv_0^2}{2} = -F s_1$ , and the braking distance is

$$s_1 = \frac{mv_0^2}{2F}.$$

*Problem 15.* The resultant  $R$  of all the forces acting on the piston in Fig. 44 changes during a certain time-interval according to the equation  $R = 0.4P(1 - kt)$ , where  $P$  is the weight of the piston,  $t$  is the time in seconds, and  $k$  a factor equal to  $1.6 \text{ sec}^{-1}$ . Determine the velocity of the piston at time  $t_1 = 0.5 \text{ sec}$ , if at time  $t_0 = 0$  it was  $v_0 = 0.2 \text{ m/sec}$ .

*Solution.* As the acting force depends on the time and the given and required quantities include  $t_1$ ,  $v_0$ , and  $v_1$ , we make use of Eq. (5.8'):

$$mv_{1x} - mv_{0x} = S_x.$$

In this case

$$S_x = \int_0^{t_1} R_x dt = 0.4P \int_0^{t_1} (1 - kt) dt = 0.4P t_1 \left(1 - \frac{k}{2} t_1\right).$$

Furthermore,  $v_{0x} = 0$ ,  $v_{1x} = v_1$  and  $m = \frac{P}{g}$ . Substituting these expressions, we obtain

$$v_1 = v_0 + 0.4gt_1 \left(1 - \frac{k}{2} t_1\right) \approx 1.4 \text{ m/sec}.$$

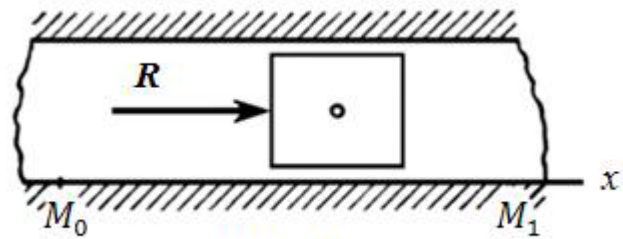


Fig. 44

*Problem 16.* A bullet of weight  $p$  fired horizontally with a velocity  $u$  hits a box of sand standing on a truck (Fig. 45). What velocity will the truck receive as a result of the impact if its weight together with the box of sand is  $P$ ?

*Solution.* Consider the bullet and the truck as one system. This enables us to exclude the forces generated when the bullet hits the sand. The sum of the projections of the external forces on the horizontal axis  $x$  is zero. Consequently,  $Q_x = \text{const.}$ , or  $Q_{0x} = Q_{1x}$ , where  $Q_0$  is the momentum of the system before the impact, and  $Q_1$ , after the impact. As the truck was motionless before the impact,

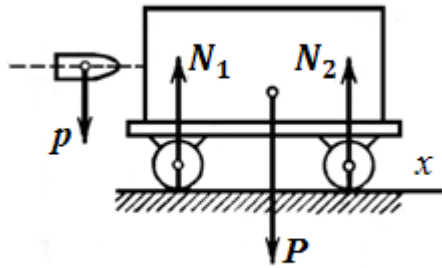


Fig. 45

$$Q_{0x} = \frac{p}{g} u.$$

After the impact the truck and bullet are moving with the same velocity  $v$ . Then

$$Q_{1x} = \frac{p+P}{g} v,$$

and equating the right sides of the expressions for  $Q_{0x}$  and  $Q_{1x}$ , we obtain

$$v = \frac{p}{p+P} u.$$

*Problem 17.* Determine the recoil of a gun if its barrel is horizontal, the weight of the recoiling parts is  $P$ , the weight of the shell is  $p$ , and the muzzle velocity of the shell is  $u$ .

*Solution.* To exclude the unknown forces developed by the pressure of the gases, consider the shell and the gun as one system.

Neglecting the resistance to the recoil during the motion of the shell in the bore,

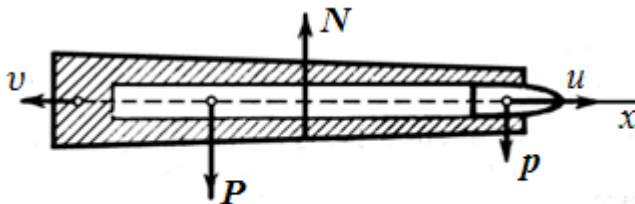


Fig. 46

we find that the sum of the projection of the applied external forces on the axis  $x$  is zero (Fig. 46). Hence,  $Q_x = \text{const.}$  and, since before the shot the system was motionless ( $Q_0 = 0$ ), for any moment of time  $Q_x = 0$ .

If the velocity of the recoiling parts at the final instant is  $v$ , then the absolute velocity of the shell at that moment is  $u + v$ . Consequently,

$$Q_x = \frac{P}{g} v_x + \frac{p}{g} (u_x + v_x) = 0,$$

whence we find

$$v_x = -\frac{p}{p+P} u_x.$$

If we knew the absolute muzzle velocity  $u_a$  of the shell, we could have substituted  $u_{ax}$  for  $u_x + v_x$ , whence

$$v_x = -\frac{p}{P} u_{ax}.$$

The minus sign in both cases means that  $v$  is in the opposite direction of  $u$ .

Note that in calculating the total momentum of a system the absolute velocity of its parts should be considered.

**Problem 18.** A jet of water of diameter  $d = 4 \text{ cm}$  is discharged from a nozzle with a velocity  $u = 10 \text{ m/sec}$  and impinges normally against a fixed vertical wall (Fig. 47). Neglecting the compression in the jet, determine the force of water on the wall.

**Solution.** To exclude the internal forces of interaction of the water particles between each other at the time of impact, apply the first of Eqs. (5.11')

$$Q_{1x} - Q_{0x} = \sum S_{kx}^e$$

to the part of the jet filling the volume  $abc$  at the given instant. Let us calculate for this volume the difference  $Q_{1x} - Q_{0x}$  for a certain time interval  $t_1$ . During this interval, the volume of water will occupy configuration  $a_1b_1c_1$ , and the value of  $Q_x$  will decrease by  $mu$ , where  $m$  is the mass of volume  $aa_1$ . The liquid filling volumes  $bb_1$  and  $cc_1$  moves normally to axis  $Ox$  and therefore does not increase the value of  $Q_x$ . As we have only  $Q_x$  decreasing,  $Q_{1x} - Q_{0x} = -mu$ . Reaction  $R$  of the wall will be the only external force acting on the given volume and giving a projection on axis  $Ox$ . Assuming  $R = \text{const.}$ , we obtain

$\sum S_{kx}^e = R_x t_1 = -R t_1$ , and  $mu = R t_1$ . Now compute  $m$ . As the displacement  $aa_1 = ut_1$ ,

$$m = \frac{\gamma \pi d^2}{g} \frac{ut_1}{4},$$

where  $\gamma$  is the weight of a unit volume, and consequently,  $\frac{\gamma}{g}$  is the mass of a unit volume of the liquid. Substituting this value and taking into account that for water  $\gamma = 10000 \text{ N/m}^3$ , we finally obtain

$$R = \frac{\gamma \pi d^2}{g} \frac{u^2}{4} = 128 \text{ N.}$$

**Problem 19.** Two discs having moments of inertia  $J_1$  and  $J_2$  are mounted on a shaft as shown in Fig. 48. If the shaft is twisted and then released, find the dependence between the angular velocities and the angle of turn of the discs in the ensuing torsional vibrations. Neglect the mass of the shaft.

**Solution.** To exclude the unknown elastic forces which cause the discs to vibrate, consider both discs and the shaft as a single system. The external forces (the reactions of the bearings and the force of gravity) intersect with axis  $x$ , whence  $\sum m_x(F_k^e) = 0$ , and  $K_x = \text{const.}$  But since at the initial moment  $K_x = 0$ , during the whole of the vibration we must have  $K_x = J_1 \omega_1 + J_2 \omega_2 = 0$  (the angular momentum of the system

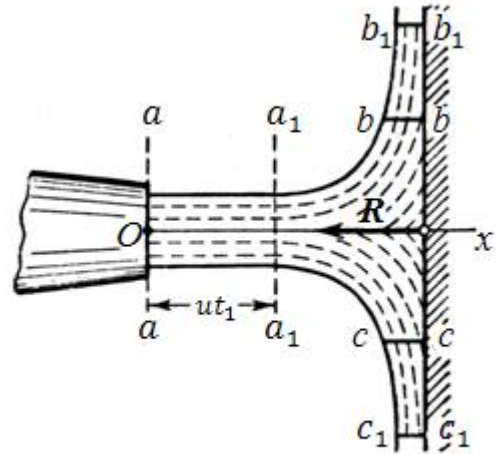


Fig. 47

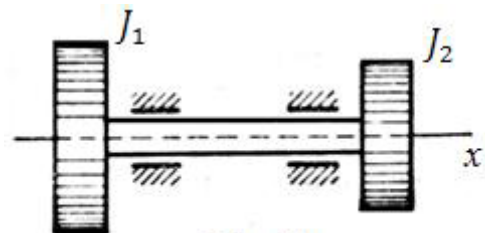


Fig. 48

with respect to the  $x$  axis equals the sum of the angular momenta of each disc with respect to the same axis). We find from this that

$$\omega_1 = \frac{J_2}{J_1} \omega_2 \quad \text{and} \quad \varphi_1 = \frac{J_2}{J_1} \varphi_2,$$

where  $\varphi_1$ , and  $\varphi_2$  are the angles through which the discs were twisted, measured from the initial position (the latter result is obtained by integrating the first equation).

Thus, the vibrations will be in opposite directions, and the angular amplitudes will be inversely proportional to the moments of inertia of the discs.

*Problem 20.* A governor  $AB$  with a moment of inertia  $J_z$  consists of two symmetrically placed weights of mass  $m$ , each attached to two springs as shown in Fig. 49, and it rotates about a vertical axis  $Oz$ . At time  $t_0 = 0$ , the governor receives an angular velocity  $\omega_0$ , and each weight starts to oscillate in damped vibration about its respective centre  $C$  at a distance  $l$  from axis  $Oz$ . Neglecting friction and considering the weights as particles, determine the dependence of the angular velocity  $\omega$  of the governor on the position of the weights.

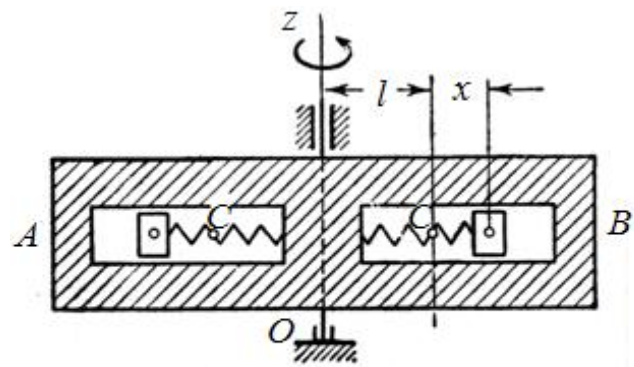


Fig. 49

*Solution.* To exclude the unknown elastic forces of the springs, consider the governor and the weights as one system. Then  $\sum m_z(\mathbf{F}_k^e) = 0$ , and  $K_z = \text{const.}$

At time  $t_0 = 0$ , the displacement  $x = 0$

and  $K_{z0} = (J_z + 2ml^2)\omega_0$ . At any arbitrary instant  $t$ ,  $K_z = [J_z + 2m(l+x)^2]\omega$ . As  $K_z = K_{z0}$ ,

$$\omega = \frac{J_z + 2ml^2}{J_z + 2m(l+x)^2} \omega_0.$$

Consequently, when  $x > 0$ ,  $\omega < \omega_0$ , and when  $x < 0$ ,  $\omega > \omega_0$ , i.e., the angular velocity changes about a mean value  $\omega_0$ . When the vibrations of the weights dampen with time,  $x$  tends to zero, and  $\omega$  to  $\omega_0$ .

*Problem 21.* A track is laid along the circumference of a disc of radius  $R$  and weight  $P$ . Standing on the track is a toy spring-wound car of weight  $p$ . The disc rotates together with the car about a vertical axis  $z$  with an angular velocity  $\omega_0$  (Fig. 50). Determine how the angular velocity of the disc will change if at some instant the car will start moving in the direction of the rotation with a velocity  $u$  relative to the disc.

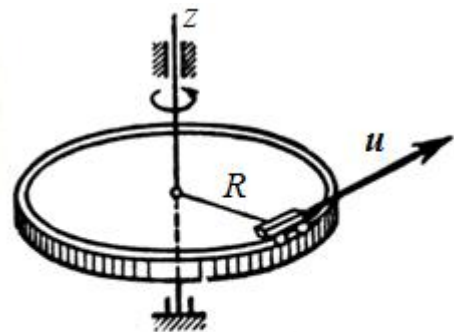


Fig. 50

*Solution.* To exclude the unknown frictional forces between the wheels of the car and the disc, consider both as one system. The moments of the external forces acting on the system with respect

to the  $z$  axis are zero. Consequently,  $K_z = \text{const.}$  Considering the disc to be homogeneous ( $J_z = 0.5MR^2$ ) and the car as a particle, we have

$$K_{z0} = \left(0.5 \frac{P}{g} R^2 + \frac{p}{g} R^2\right) \omega_0.$$

When the car starts moving its absolute velocity will be  $v_a = u + \omega R$ , where  $\omega$  is the new angular velocity of the disc. The angular momentum of the car about the  $z$  axis will be  $mv_a R = m(uR + \omega R^2)$ , and for the whole system we have

$$K_{z1} = 0.5 \frac{P}{g} R^2 \omega + \frac{p}{g} (uR + R^2 \omega).$$

As  $K_z = \text{const.}$ ,  $K_{z1} = K_{z0}$ , whence

$$\omega = \omega_0 - \frac{P}{0.5P + p} \frac{u}{R}.$$

The angular velocity of the disc, we see, decreases. If the car travels in the opposite direction,  $\omega$  will increase.

Note that in calculating  $K_z$  the absolute velocities of all moving points of the system were taken.

*Problem 22.* Wound on a drum of weight  $P$  and radius  $r$  (Fig. 51) is a string carrying a load  $A$  of weight  $Q$ . Neglecting the mass of the string and friction, determine the angular acceleration of the drum when the load falls, if the radius of gyration of the drum with respect to its axis is  $\rho$ .

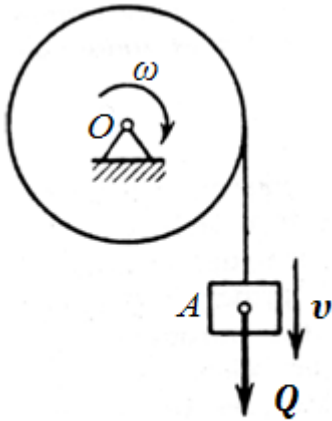


Fig. 51

*Solution.* Applying the theorem of moments with respect to axis  $O$ , we have

$$\frac{dK_0}{dt} = \sum m_0 (\mathbf{F}_k^e).$$

The moving system consists of two bodies, consequently

$$K_0 = K_{\text{drum}} + K_{\text{load}}.$$

The load is in translational motion, and we consider it as a particle. Its velocity is  $v = \omega r$ . The drum rotates about a fixed axis, consequently,

$$K_{\text{load}} = \frac{Q}{g} vr = \frac{Q}{g} r^2 \omega; \quad K_{\text{drum}} = J_0 \omega = \frac{P}{g} \rho^2 \omega,$$

and

$$K_0 = (Qr^2 + P\rho^2) \frac{\omega}{g}.$$

Substituting this expression for  $K_0$ , we obtain

$$\frac{Qr^2 + P\rho^2}{g} \frac{d\omega}{dt} = Qr,$$

whence

$$\varepsilon = \frac{Qrg}{Qr^2 + P_Q^2}$$

**Problem 23.** A weight attached to a string of length  $l$  (Fig. 52b) is displaced from the vertical at an angle  $\varphi_0$  and released from rest. Determine the velocity of the weight at the instant when the thread makes an angle  $\varphi$  with the vertical.

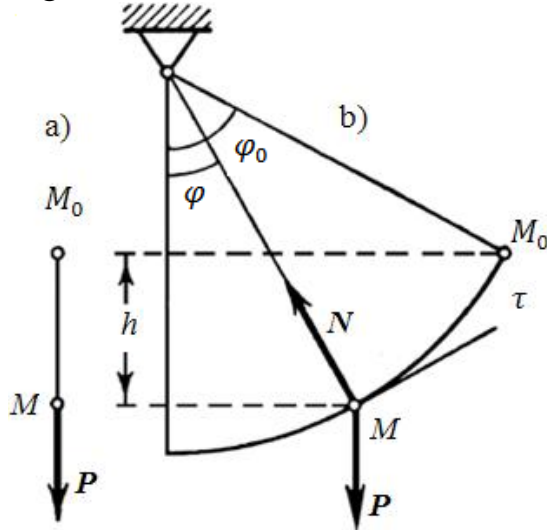


Fig. 52

**Solution.** As the conditions of the problem include the displacement of the weight, defined by the angle through which the thread passes, and the velocities  $v_0$  and  $v_1$ , we make use of the theorem of the change in kinetic energy:

$$\frac{mv_1^2}{2} - \frac{mv_0^2}{2} = \sum A_k$$

Acting on the weight is the force of gravity  $P$  and the reaction of the thread  $N$ . The work done by force  $N$  is zero, as  $N_\tau = 0$ . For force  $P$  we have  $A(P) = Ph = mgh$ .

As  $v_0 = 0$ , we obtain  $\frac{mv_1^2}{2} = mgh$ , whence

$$v_1 = \sqrt{2gh}$$

Evidently, the result is the same for the velocity  $v$  of a freely falling weight (Fig. 52a).

In our problem  $h = l \cos \varphi - l \cos \varphi_0$ , and finally

$$v = \sqrt{2gl(\cos \varphi - \cos \varphi_0)}$$

**Problem 24.** The length  $l_0$  of an uncompressed valve spring is 6 cm. When the valve is completely open it is lifted to a height  $s = 0.6$  cm and the length of the compressed spring is  $l = 4$  cm (Fig. 53). The stiffness of the spring is  $c = 1$  N/cm and the weight of the valve is  $p = 4$  N. Neglecting the gravitational and resisting forces, determine the velocity of the valve at the moment of its closure.

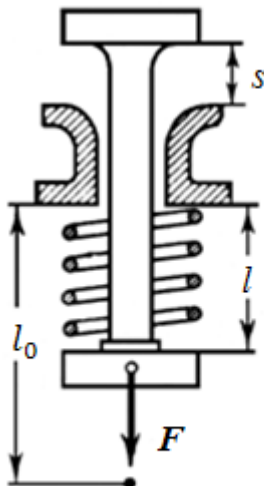


Fig. 53

**Solution.** The elastic force  $F$  acting on the valve depends on the displacement  $s$  of the valve, which is given. Therefore we use Eq. (5.36):

$$\frac{mv_1^2}{2} - \frac{mv_0^2}{2} = A_k$$

According to the conditions of the problem, the only force doing work is the elastic force of the spring. Hence, we have

$$A_k = \frac{c}{2} [(\Delta l_{in})^2 - (\Delta l_{fin})^2]$$

In our case  $\Delta l_{in} = l_0 - l = 2$  cm,  $\Delta l_{fin} = l_0 - l - s = 1.4$  cm.

Furthermore,  $v_0 = 0$  and  $m = \frac{P}{g}$ . Substituting these expressions, we obtain

$$v_1 = \sqrt{\frac{cg}{p}(\Delta l_{in})^2 - (\Delta l_{fin})^2} \approx 0.22 \text{ m/sec.}$$

*Problem 25.* An elastic beam supporting a weight at the centre of its span (Fig. 54) deflects by an amount  $\delta_{st}$  (the static deflection of the beam). Neglecting the weight of the beam, determine its maximum deflection  $\delta_{max}$  if the weight is dropped on it from a height  $H$ .

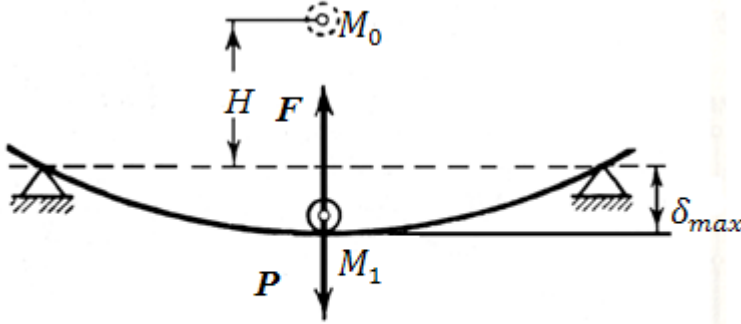


Fig. 54

*Solution.* As in the previous problem, we apply Eq. (5.36). The initial velocity  $v_0$  and the final velocity  $v_1$  (at the instant of the maximum deflection of the beam) of the weight are each zero, and Eq. (5.36) takes the form

$$\sum A_k = 0.$$

The forces doing work are the gravitational force  $P$  in the displacement  $M_0M_1$  and the elastic force  $F$  of the beam in the deflection of the beam.

Also,  $A(P) = P(H + \delta_{max})$ ,  $A(F) = \frac{c}{2}\delta_{max}^2$ , as for the beam  $\Delta l_{in} = 0$ ,  $\Delta l_{fin} = \delta_{max}$ . Substituting these expressions, we obtain

$$P(H + \delta_{max}) - \frac{c}{2}\delta_{max}^2 = 0.$$

When the weight on the beam is in equilibrium it is balanced by elastic force. Consequently,  $P = c\delta_{st}$ , and the last equation can be written in the form

$$\delta_{max}^2 - 2\delta_{max}\delta_{st} - 2\delta_{st}H = 0.$$

Solving this quadratic equation, and taking into account that according to the conditions of the problem  $\delta_{max} > 0$ , we find

$$\delta_{max} = \delta_{st} + \sqrt{\delta_{st}^2 + 2H\delta_{st}}.$$

*Problem 26.* A load of weight  $P$  attached to a string of length  $l$  is displaced through an angle  $\alpha$  from the vertical to a position  $M_0$  and released from rest (Fig. 55). Determine the tension in the thread when the load is in its lowest position  $M_1$ .

*Solution.* Draw the load in the position for which the tension in the string has to be found, i.e., in position  $M_1$ . Acting on the load is its weight  $P$  and the reaction of the thread  $T$ . Draw the inward normal  $M_1n$  and write Eq. (1.2), taking into account that  $a = a_n = \frac{v^2}{\rho}$  and in the present case  $\rho = l$ . We have:

$$\frac{mv_1^2}{l} = T - P \text{ or } T = P + \frac{mv_1^2}{l},$$

where  $v_1$  is the velocity of the load at position  $M_1$ . To determine  $v_1$ , we make use of Eq. (5.36'):

$$\frac{mv_1^2}{2} - \frac{mv_0^2}{2} = A_{(M_0M_1)}^a.$$

On the section  $M_0M_1$  only force  $P$  does any work. Therefore,  $A^a = Ph = Pl(1 - \cos \alpha)$ .

As  $v_0 = 0$ , substituting the expression developed for the work, we obtain  $mv_1^2 = 2Pl(1 - \cos \alpha)$ , and finally

$$T = P(3 - 2\cos \alpha).$$

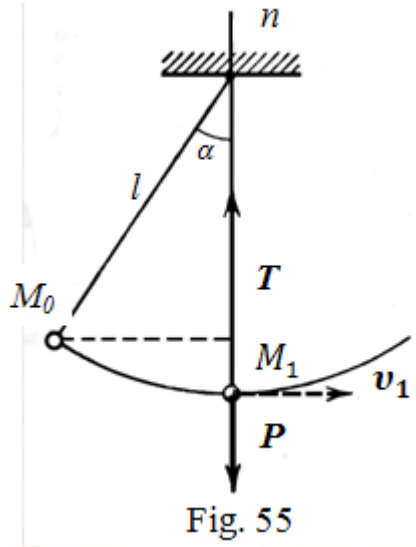


Fig. 55

In the special case, when the initial angle of deflection is  $90^\circ$ , the tension in the string when it is in vertical position will be  $3P$ , i.e., treble the weight of the load. The result shows that dynamic reactions can differ considerably from static reactions.

*Problem 27.* A grooved track makes two circular arcs  $AB$  and  $BD$  of radius  $R$  in a vertical plane; the tangent  $BE$  through their point of conjugation is horizontal (Fig. 56).

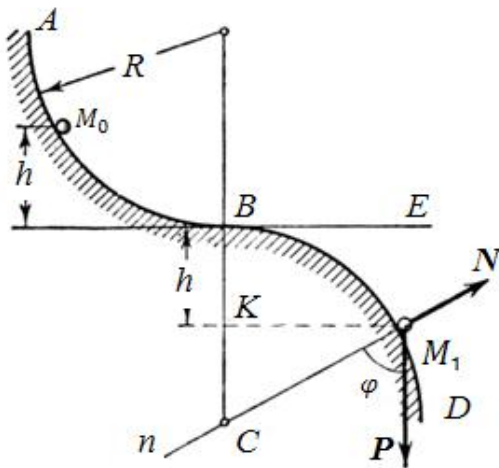


Fig. 56

Neglecting friction, determine the height  $h$  from  $BE$  at which a heavy ball should be placed on the track so that it would shoot out of the track at point  $M_1$  lying at the same distance  $h$  below  $BE$ .

*Solution.* The ball will leave the track at a point  $M_1$  such that its pressure on the track (or the reaction  $N$  of the track) is zero. Consequently, our problem is reduced to the determination of  $N$ . Draw the ball at  $M_1$ . Acting on it are the force of gravity  $P$  and the reaction of the track  $N$ . Writing Eq. (5.36') for the

projections on the inward normal  $M_1n$ , we have

$$\frac{mv_1^2}{R} = P \cos \varphi - N.$$

Since at the point of departure  $N = 0$ , and taking into account that  $R \cos \varphi = KC = R - h$ , for determining  $h$  we obtain equation

$$mv_1^2 = P(R - h).$$

The value of  $mv_1^2$  can be found from the theorem of the change in kinetic energy. As  $v_0 = 0$ , Eq. (5.36') gives

$$\frac{mv_1^2}{2} = A_{(M_0M_1)}^a.$$



The only force that does work is  $\mathbf{P}$ , and  $A(\mathbf{P}) = P2h$ . Consequently,  $mv_1^2 = 4Ph$ . Substituting this expression of  $mv_1^2$ , we obtain  $4h = R - h$ , whence  $h = 0.2R$ .

*Problem 28.* A load  $M$  is attached to a string of length  $l$  (Fig. 57). What is the least initial velocity  $v_0$  perpendicular to the string that should be imparted to the load for it to describe a complete circle?

*Solution.* The load will describe a complete circle if nowhere along its path (except, possibly, point  $M'$ ) will the tension in the string become zero, i.e., if the string remains taut. If on the other hand, at any point  $M_1$  where  $v_1 \neq 0$  the tension becomes zero, the string will no longer constrain the load, which will continue to move as a free body.

To solve the problem we must determine the tension  $T$  in the string at any point  $M$  defined by angle  $\varphi$  and then require that  $T > 0$  for any angle  $\varphi \neq 180^\circ$ .

Acting on the load at point  $M$  are its weight  $\mathbf{P}$  and the tension of the thread  $\mathbf{T}$ . Writing Eq. (1.2) for the projections on the inward normal  $Mn$ , we obtain

$$\frac{mv^2}{l} = T - P \cos \varphi ,$$

where  $v$  is the velocity of the load at point  $M$ . To determine  $v$  we apply the theorem of the change in kinetic energy:

$$\frac{mv^2}{2} - \frac{mv_0^2}{2} = A_{(M_0M_1)}^a.$$

In our case  $A_{(M_0M_1)}^a = -Ph = -Pl(1 - \cos \varphi)$ , and consequently  $mv^2 = mv_0^2 - 2Pl(1 - \cos \varphi)$ .

Substituting this expression of  $mv^2$  and solving for  $T$ , we obtain

$$T = P \left( \frac{v_0^2}{gl} - 2 + 3 \cos \varphi \right).$$

The least value of  $T$  is at  $\varphi = 180^\circ$ :

$$T_{\min} = P \left( \frac{v_0^2}{gl} - 5 \right).$$

The condition for  $T$  never to become zero (except, possibly, at point  $M'$ ) is that  $T_{\min} \geq 0$ . Hence,

$$\frac{v_0^2}{gl} \geq 5 \quad \text{or} \quad v_0 \geq \sqrt{5gl}.$$

Thus, the least initial velocity at which the load will describe a complete circle is given by the equation

$$v_{0 \min} = \sqrt{5gl}.$$

Let us assume that the load is attached not to a thread but to a rigid light (weightless) rod of length  $l$ . In this case (since, unlike a thread, a rod can work both in tension and in compression) the load will describe a complete circle if the velocity does not become zero anywhere (except, possibly, at point  $M'$ ). Applying Eq. (5.36') for the displacement  $M_0M'$  and assuming  $v = 0$  at point  $M'$ , we obtain

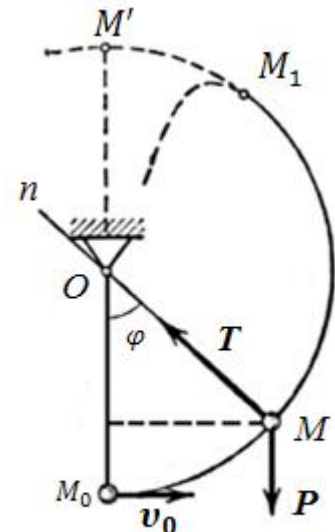


Fig. 57

$$\frac{-mv_0^2}{2} = -mg2l.$$

Hence  $v_{0\min} = \sqrt{4gl}$ .

*Problem 29.* Find the kinetic energy of a uniform cylindrical wheel of mass  $M$  rolling without slipping, if the velocity of its centre is  $v_c$ .

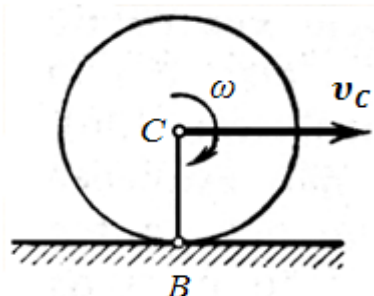


Fig. 58

*Solution.* The wheel is in plane motion. By Eq. (5.24),

$$T = \frac{1}{2}Mv_c^2 + \frac{1}{2}J_C\omega^2.$$

As the body is an uniform cylinder, we have  $J_C = 0.5MR^2$ , where  $R$  is the radius of the wheel. On the other hand, since point  $B$  is the instantaneous centre of zero velocity of the wheel (Fig. 58),  $v_c = \omega BC = \omega R$ , whence  $\omega = v_c/R$ .

Substituting these expressions, we find

$$T = \frac{1}{2}Mv_c^2 + \frac{1}{4}MR^2 \frac{v_c^2}{R^2} = \frac{3}{4}Mv_c^2.$$

*Problem 30.* When body  $A$  in Fig. 59 translates with a velocity  $\mathbf{u}$ , body  $B$  moves in the slots of body  $A$  with a velocity  $\mathbf{v}$ . If angle  $\varphi$  is known, determine the kinetic energy of body  $B$ .

*Solution.* The absolute motion of body  $B$  is a translation with a velocity  $\mathbf{v}_a = \mathbf{u} + \mathbf{v}$ . Then

$$T = \frac{1}{2}Mv_a^2 = \frac{1}{2}M(u^2 + v^2 + 2uv \cos \alpha).$$

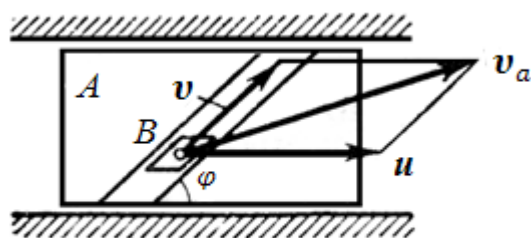


Fig. 59

A characteristic mistake in problems of this type is to regard the kinetic energy of the body as the sum of the energies of the relative and transport motions:

$$T = T_{rel} + T_{tr} = \frac{1}{2}Mv^2 + \frac{1}{2}Mu^2,$$

and thus, it will be noticed, the component  $Muv \cos \alpha$  drops out of the solution.

Thus, in the most general case of relative motion, the total kinetic energy of a body does not equal the sum of the kinetic energies of its relative and transport motions.

*Problem 31.* A mechanism consists of a part which is translated with a velocity  $\mathbf{u}$ , and a rod  $AB$  of length  $l$  and mass  $M$  hinged at  $A$  (Fig. 60). The rod rotates about axis  $A$  with an angular velocity  $\omega$ . Determine the kinetic energy of the rod if angle  $\varphi$  is given.

*Solution.* The rod performs plane motion, and by Eq. (5.24)

$$T = \frac{1}{2} M v_C^2 + \frac{1}{2} J_C \omega^2.$$

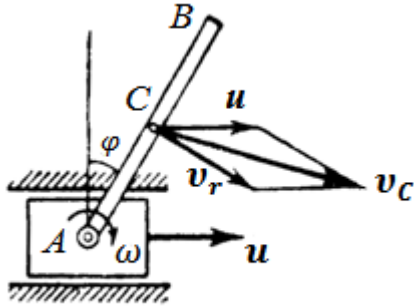


Fig. 60

The velocity of point C is compounded of the velocities  $u$  and  $v_r$ , where in magnitude  $v_r = \omega \frac{l}{2}$ . Consequently,  $v_C^2 = u^2 + v_r^2 + 2uv_r \cos \varphi$ .

The angular velocity of the rod about C is the same as about A, as  $\omega$  does not depend on the location of the pole. Taking into account that  $J_C = \frac{Ml^2}{12}$  and substituting all these expressions, we obtain

$$\begin{aligned} T &= \frac{1}{2} M \left( u^2 + \omega^2 \frac{l^2}{4} + u\omega l \cos \varphi \right) + \frac{1}{24} M l^2 \omega^2 = \\ &= \frac{1}{2} M u^2 + \frac{1}{6} M l^2 \omega^2 + \frac{1}{2} M l \omega u \cos \varphi. \end{aligned}$$

*Problem 32.* A rod AB of length  $l$  is hinged as shown at point A (Fig. 61). Neglecting friction, determine the minimum velocity  $\omega_0$  that must be imparted to the rod so that it would swing into a horizontal configuration.

*Solution.* The given and required quantities include  $\omega_0$ ,  $\omega_1 = 0$ , and the displacement of the system as defined by angle  $B_0AB_1$ . Therefore, the problem is best solved by applying Eq. (5.39):

$$T_1 - T_0 = \sum A_k^e.$$

Denoting the weight of the rod by  $P$ , compute all the quantities in equation. From Eqs. (5.23) and taking into account that  $J_A = \frac{Ml^2}{3}$ , we find

$$T_0 = \frac{1}{2} J_A \omega_0^2 = \frac{1}{6} \frac{P}{g} l^2 \omega_0^2.$$

Since in the final configuration the velocity of the rod is zero,  $T_1 = 0$ . The work is done only by force  $P$ , and  $A^e = -Ph_C = -P \frac{l}{2}$ . Substituting these values, we obtain

$$-\frac{1}{6} \frac{P}{g} l^2 \omega_0^2 = -P \frac{l}{2},$$

whence  $\omega_0 = \sqrt{\frac{3g}{l}}$ .

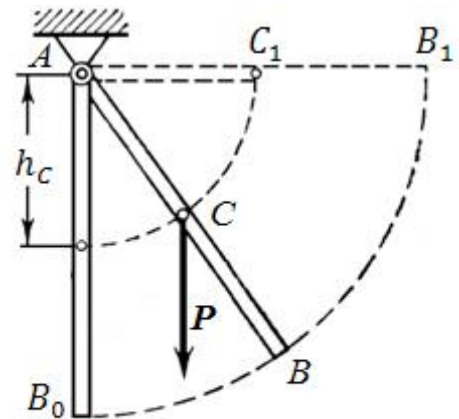


Fig. 61

*Problem 33.* Two pulleys A and B are connected by a belt (Fig. 62). When the motor is switched off, pulley A of radius  $R$ , has an angular velocity  $\omega_0$ . The total weight of the pulleys is  $P$ , and of the belt  $p$ . A brake shoe is applied to pulley A with a force  $Q$  to stop the rotation; the coefficient of friction of the shoe on the pulley is  $f$ .

Neglecting friction in the axles and considering the pulleys to be homogeneous discs, determine how many revolutions pulley  $A$  will make before stopping.

*Solution.* We shall use Eq. (5.39) to determine the required number of revolutions  $N$ :

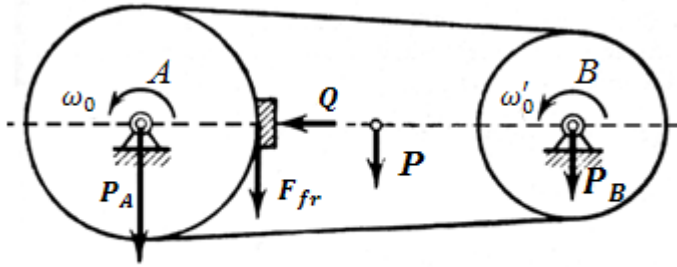


Fig. 62

$$T_1 - T_0 = \sum A_k^e.$$

From the conditions of the problem  $T_1 = 0$ , and  $T_0 = T_A + T_B + T_b$ . Taking into account that the initial velocities of all the points of the belt  $v_{b0} = \omega_0 R = \omega'_0 r$ , where  $\omega'_0$  and  $r$  are the initial

angular velocity and the radius of pulley  $B$ , we find:

$$T_A = \frac{1}{2} \left( \frac{P_A}{2g} R^2 \right) \omega_0^2, \quad T_B = \frac{1}{2} \left( \frac{P_B}{2g} r^2 \right) \omega_0'^2 = \frac{1}{4} \frac{P_B}{g} R^2 \omega_0^2,$$

$$T_b = \frac{1}{2} \frac{p}{g} v_{b0}^2 = \frac{1}{2} \frac{p}{g} R^2 \omega_0^2.$$

The last equation follows from the fact that all the points of the belt move with the same speed. Finally, as  $P_A + P_B = P$ , we obtain

$$T_0 = \frac{P + 2p}{4g} R^2 \omega_0^2.$$

Now compute the work done by the forces. In this case, the work done by gravity is zero, as the centers of gravity of the pulleys and the belt do not change their position during the motion. The force of friction  $F_{fr} = fQ$ , and the work done by it is found from Eq. (5.32):

$$A_{fr} = -(fQR)\varphi_1 = -fQR \cdot 2\pi N.$$

Substituting all the found values, we obtain finally

$$N = \frac{(P + 2p)R\omega_0^2}{8\pi g f Q}.$$

*Problem 34.* A cart is drawn with a constant force  $Q = 160 \text{ N}$  up an inclined plane making an angle  $\alpha = 30^\circ$  with the horizontal (Fig. 63). The platform of the cart weighs  $P = 180 \text{ N}$ , and each of its uniform wheels weighs  $p = 20 \text{ N}$ . Determine: 1) the linear velocity  $v_1$  of the cart, when it has travelled a distance  $l = 4 \text{ m}$ , if  $v_0 = 0$ ; 2) the acceleration of the cart. The wheels roll without slipping. Neglect the rolling friction.

*Solution.* 1) Let us use Eq. (5.39) to determine  $v_1$ :

$$T_1 - T_0 = \sum A_k^e.$$

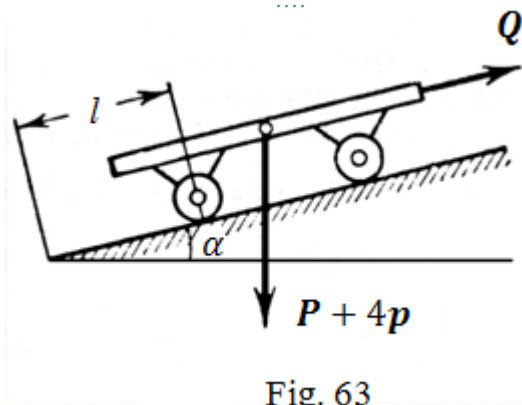


Fig. 63

In our case  $T_0 = 0$  and  $T_0 = T_{platform} + 4T_{wheel}$ . The cart is in translatory motion, and the kinetic energy of a uniform rolling wheel was calculated in Problem 29. Thus,

$$T_1 = \frac{1}{2} \frac{P}{g} v_1^2 + 4 \left( \frac{3}{4} \frac{p}{g} v_1^2 \right) = \frac{1}{2g} (P + 6p) v_1^2.$$

Work is done by force  $Q$  and the force of gravity  $P_1 = P + 4p$ . The work done by the frictional forces preventing slippage

and by the normal reactions is zero. Making the necessary computations, we find

$$A(Q) = Ql; \quad A(P_1) = -(P + 4p)h_C = -(P + 4p)l \sin \alpha.$$

Substituting these expressions, we obtain

$$\frac{1}{2g} (P + 6p) v_1^2 = [Q - (P + 4p) \sin \alpha] l,$$

whence

$$v_1 = \sqrt{\frac{2gl[Q - (P + 4p) \sin \alpha]}{P + 6p}} = 2.8 \text{ m/sec.}$$

2) To determine the acceleration  $a$ , let us consider the quantities  $v_1$  and  $l$  as variables. Then, differentiating through with respect to time, we have

$$\frac{1}{g} (P + 6p) v \frac{dv}{dt} = [Q - (P + 4p) \sin \alpha] \frac{dl}{dt}.$$

But  $\frac{dl}{dt} = v$ , and  $\frac{dv}{dt} = a$  and, eliminating  $v$ , we have

$$a = \frac{Q - (P + 4p) \sin \alpha}{P + 6p} g = 0.98 \text{ m/sec}^2.$$

*Problem 35.* One end of a string passing over a pulley  $O$  (Fig. 64) is wound on a cylinder of radius  $R$  and weight  $P$ ; attached to the other end is a load  $D$  of weight  $Q$ . If  $v_{C0} = 0$ , determine the velocity  $v_C$  of the centre  $C$  of the cylinder after it has travelled a distance  $s$ , and the acceleration  $a_C$  of the centre. The coefficient of rolling friction of the cylinder is  $k$ , the radius of gyration of the cylinder with respect to its axis is  $\rho$ . Neglect the mass of the string and the pulley

*Solution.* 1) We use Eq. (5.39) to determine the velocity  $v_C$ :

$$T - T_0 = \sum A_k^e.$$

In our case  $T_0 = 0$  and  $T = T_{cyl} + T_D$ . From Eqs. (4.7), (5.22), and (5.24),

$$T_D = \frac{1}{2} \frac{Q}{g} v_D^2, \quad T_{cyl} = \frac{1}{2} \frac{P}{g} v_C^2 + \frac{1}{2} \left( \frac{P}{g} \rho^2 \right) \omega^2.$$

As point  $B$  is the instantaneous centre of zero velocity,

$$\omega = \frac{v_C}{R}, \quad \text{and} \quad v_D = v_A = 2v_C.$$

Consequently,

$$T = \frac{1}{2g} \left[ 4Q + P \left( 1 + \frac{\rho^2}{R^2} \right) \right] v_C^2.$$

The forces doing work are  $Q$  and the couple  $(N, P)$ . As  $v_D = 2v_C$ , the displacement of load  $D$  is  $h = 2s$ , and  $A(Q) = Q \cdot 2s$ . The work done by the forces opposing the rolling can be found from Eq. (5.34), as  $N = P = \text{const}$ . Then,

$$\sum A_k^e = 2Qs - \frac{k}{R} Ps.$$

Substituting the found expressions, we obtain

$$\frac{1}{2g} \left[ 4Q + P \left( 1 + \frac{\rho^2}{R^2} \right) \right] v_C^2 = \left( 2Q - \frac{k}{R} P \right) s,$$

whence

$$v_C = \sqrt{\frac{2g(2QR - kP)Rs}{4QR^2 + P(R^2 + \rho^2)}}.$$

2) As in the preceding problem, to determine  $a_C$  differentiate both sides of the last equation with respect to time. Taking into account that  $\frac{ds}{dt} = v_C$ , we finally obtain

$$a_C = \frac{(2QR - kP)R}{4QR^2 + P(R^2 + \rho^2)} g.$$

*Problem 36.* Referring to Fig. 65, a spiral spring is attached to gear 1 of radius  $r$  and weight  $P$  and to crank  $OC$  of length  $l$  and weight  $Q$  on which the gear is mounted. Gear 1 rolls on a fixed gear 2 of radius  $R = l - r$ . The moment of the spring  $M_{sp} = c\alpha$ , where  $\alpha$  is the angle of rotation of gear 1 with respect to the crank. Neglecting friction, determine the period of vibration of the crank if it is disturbed from its position of equilibrium. The mechanism works in a horizontal plane.

*Solution.* We shall define the position of the crank by the angle  $\varphi$  measured from its equilibrium position. To exclude the unknown reaction of axis  $C$  from the computation, consider gear 1 and the crank as a single system and develop the differential equation of its motion from Eq. (5.38).

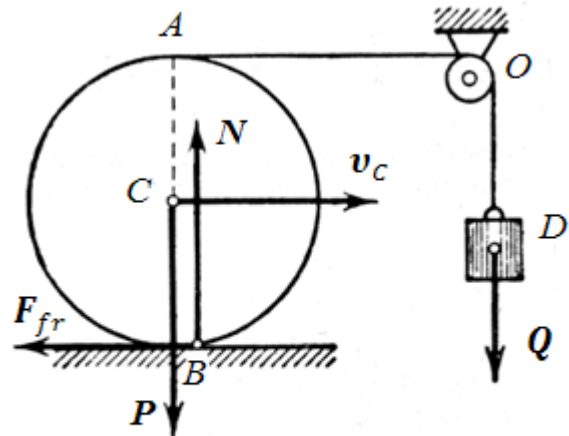


Fig. 64

First, compute the kinetic energy  $T$  of the system in terms of the angular velocity  $\omega_{cr}$  of the crank (as we are developing the law of motion of the crank). We have:

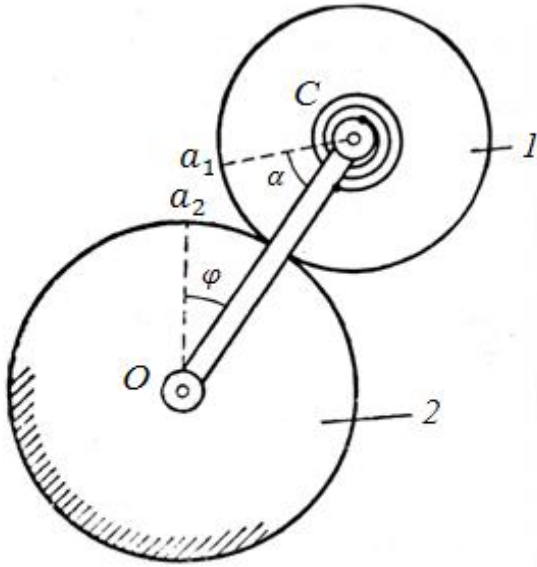


Fig. 65

$$T = T_{cr} + T_{gear} = \frac{1}{2}J_{Ocr}\omega_{cr}^2 + \frac{1}{2}\frac{P}{g}v_C^2 + \frac{1}{2}J_{Cgear}\omega_{gear}^2.$$

Considering the crank as a homogeneous rod and the gear as a uniform disc, and taking into account that the point of contact is the instantaneous centre of zero velocity of gear 1, we have

$$J_{Ocr} = \frac{1}{3}\frac{Q}{g}l^2, \quad J_{Cgear} = \frac{1}{2}\frac{P}{g}r^2, \\ v_C = \omega_{cr}l, \quad \omega_{gear} = \frac{v_C}{r} = \frac{l}{r}\omega_{cr}.$$

Note again that Eq. (5.24) which is used to compute  $T_{gear}$ , contains the

absolute angular velocity of the gear, not its relative velocity of rotation with respect to the crank. Substituting all the determined quantities, we finally obtain

$$T = \frac{1}{12g}(2Q + 9P)l^2\omega_{cr}^2.$$

Now let us compute the elemental work. In this case no external forces do any work, therefore,  $dA^e = 0$ . The elemental work done by the elastic force of the spring (an internal force) in turning the gear through an angle  $\alpha$  about the crank is  $dA^i = -M_{sp}d\alpha = -cada\alpha$  (the minus sign indicates that the moment is directed opposite the direction through which the gear is turned). As we are seeking the law of motion of the crank, let us express angle  $\alpha$  in terms of  $\varphi$ . As  $R\varphi = r\alpha$ ,

$$\frac{\alpha}{R} = \frac{\varphi}{r}, \quad \alpha = \frac{l-r}{r}\varphi \quad \text{and} \quad dA^i = -c\frac{(l-r)^2}{r^2}\varphi d\varphi.$$

Writing now the equation  $dT = dA^i$ , we have

$$\frac{1}{6g}(2Q + 9P)l^2\omega_{cr} \cdot d\omega_{cr} = -c\frac{(l-r)^2}{r^2}\varphi d\varphi.$$

Dividing through by  $dt$  and taking into account that  $\frac{d\varphi}{dt} = \omega_{cr}$  and  $\frac{d\omega_{cr}}{dt} = \frac{d^2\varphi}{dt^2}$ , we finally obtain the differential equation of motion of the system in the form

$$\frac{d^2\varphi}{dt^2} + k^2\varphi = 0,$$

where

$$k^2 = \frac{6gc(l-r)^2}{(2Q+9P)l^2r^2}.$$

This equation is a differential equation of harmonic motion. When moved from its equilibrium position the crank will perform simple harmonic motion the period of which will be

$$T = \frac{2\pi}{k} = 2\pi \frac{lr}{l-r} \sqrt{\frac{2Q+9P}{6gc}}.$$

*Problem 37.* A wheel of radius  $R$  and weight  $P$  rotates on its axis  $O$  with an angular velocity  $\omega_0$  (Fig. 66). A brake shoe is applied to the wheel at some instant with a force  $Q$ . The coefficient of friction of the shoe on the wheel is  $f$ . Neglecting friction in the axle and the weight of the spokes, determine in how many seconds the wheel will stop.

*Solution.* Taking into account Eqs. (5.18), (5.20) and considering the moment positive in the direction of the rotation, we have

$$J_O \frac{d\omega}{dt} = -fQr,$$

as the force of friction  $F_{fr} = fQ$ . From this, integrating, we obtain

$$J_O \omega = -fQrt + C_1.$$

According to the initial conditions, at  $t = 0$ ,  $\omega = \omega_0$ , consequently  $C_1 = J_O \omega_0$ , and finally

$$\omega = \omega_0 - \frac{fQr}{J_O} t.$$

At  $t = t_1$ , when the wheel stops,  $\omega = 0$ . Substituting this expression and taking into account that for the rim (a ring)  $J_O = \frac{P}{g} r^2$ , we obtain

$$t_1 = \frac{J_O \omega_0}{fQr} = \frac{Pr\omega_0}{fQg}.$$

If we want to determine the number of revolutions of the wheel until it stops, it is more convenient to apply the theorem of the change in kinetic energy.

*Problem 38.* A vertical cylindrical rotor whose moment of inertia with respect to its axis is  $J_z$  (Fig. 67) is made to revolve by an applied torque  $M_t$ . Determine how the angular velocity  $\omega$  of the rotor will change during the motion if  $\omega_0 = 0$  and the moment of the resisting force of the air is proportional to  $\omega$ , i.e.,  $M_{res} = \mu\omega$ .

*Solution.* The differential equation of the rotation of the rotor has the form (assuming the moments in the direction of rotation to be positive)

$$J_z \frac{d\omega}{dt} = M_t - \mu\omega.$$

Separating the variables and assuming  $\frac{\mu}{J_z} = n$ , we have

$$-\mu \frac{d\omega}{M_t - \mu\omega} = -ndt,$$

whence, integrating, we find

$$\ln(M_t - \mu\omega) = -nt + \ln C.$$

As, at  $t = 0$ ,  $\omega = 0$ , then  $C = M_t$ , and

$$\ln \frac{M_t - \mu\omega}{M_t} = -nt,$$

or

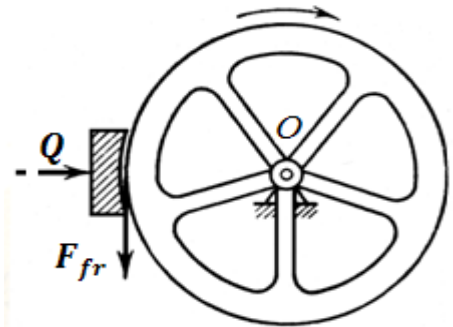


Fig. 66

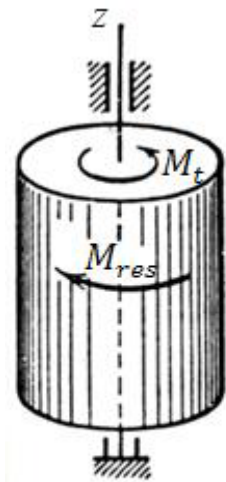


Fig. 67



$$\frac{M_t - \mu\omega}{M_t} = e^{-nt},$$

and finally we obtain

$$\omega = \frac{M_t}{\mu} (1 - e^{-nt}).$$

The angular velocity of the rotor increases with time and tends towards the limiting value  $a_{lim} = \frac{M_t}{\mu}$ .

*Problem 39.* A uniform circular cylinder rolls down an inclined plane without slipping (Fig. 68). Neglecting rolling friction, determine the acceleration of the centre of the cylinder and the limiting impending slip.

*Solution.* Let us introduce the following notations:  $\alpha$  for the angle of inclination of the surface,  $P$  for the weight of the cylinder,  $R$  for its radius, and  $F$  for the limiting friction of impending slip. Let us also direct the  $x$  axis along the inclined plane and the  $y$  axis perpendicular to it.

As the centre of mass of the cylinder does not move parallel to the  $y$  axis,  $a_{cy} = 0$ , and the sum of the projections of all the forces on the  $y$  axis is also zero. Thus,  $N = P \cos \alpha$ .

Take into account that  $a_{cx} = a_c$ . Neglecting rolling friction and taking the direction of rotation of the cylinder as the positive direction of the moment of force, we find

$$Ma_c = P \sin \alpha - F, \quad J_c \varepsilon = FR.$$

These equations contain three unknown quantities  $J_c$ ,  $\varepsilon$  and  $F$  (we cannot consider  $F = fN$  here, because this equality is valid only when the point of contact slides on the surface; when there is no sliding it is possible for  $F \leq fN$ ). We obtain an additional relationship between the unknown quantities if we take into account that in pure rolling  $v_c = \omega R$ , whence, differentiating, we have  $a_c = \varepsilon R$ . For uniform cylinder  $J_c = 0,5MR^2$ , then the second of equations takes the form

$$\frac{1}{2}Ma_c = F.$$

Substituting this expression of  $F$  into the first of equations, we obtain  $a_c = \frac{2}{3}g \sin \alpha$ .

Now we find  $F = \frac{1}{3}P \sin \alpha$ .

This is the friction force that must act on the rolling cylinder if it is not to slip. It was pointed out before that  $F = fN$ . Consequently, pure rolling takes place when

$$\frac{1}{3}P \sin \alpha \leq fP \cos \alpha \quad \text{or} \quad f \geq \frac{1}{3} \tan \alpha.$$

If the coefficient of friction is less than this, force  $F$  cannot attain the obtained value, and the cylinder will slip. In this case  $v_c$  and  $\omega$  are not related by the equality  $v_c = \omega R$  (the point of contact is not the instantaneous centre of zero velocity), but

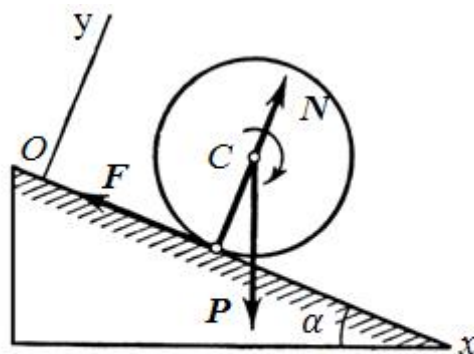


Fig. 68

now  $F$  has a limiting value, i.e.,  $fN = fP \cos \alpha$ , and equations take the form

$$\frac{P}{g} a_c = P(\sin \alpha - f \cos \alpha), \quad \frac{1}{2} \frac{P}{g} R^2 \varepsilon = fPR \cos \alpha,$$

whence

$$a_c = g(\sin \alpha - f \cos \alpha), \quad \varepsilon = \frac{2gf}{R} \cos \alpha.$$

In this case the centre of the cylinder moves with acceleration  $a_c$ , while the cylinder itself rotates with an angular acceleration  $\varepsilon$ .

*Problem 40.* Solve the previous problem taking into account the resistance to rolling, assuming the coefficient of rolling friction to be  $k$ .

*Solution.* In order to give an example of another method of computation, let us find  $a_c$  with the help of the theorem of the change in kinetic energy, i.e., the equation

$$dT = dA^e.$$

In our case,

$$T = \frac{3}{4} M v_C^2.$$

Only the force  $\mathbf{P}$  and the resisting moment perform any work. The work done by forces  $\mathbf{F}$  and  $\mathbf{N}$  in rolling is zero. Then, taking into account Eq. (5.33), we obtain (see Fig. 68, but now with force  $\mathbf{N}$  shifted by the value  $k$  in the direction of the motion):

$$dA^e = P \sin \alpha \cdot ds_C - \frac{k}{R} N ds_C = P \left( \sin \alpha - \frac{k}{R} \cos \alpha \right) dS_C.$$

Substituting the determined quantities and dividing through by  $dt$ , we have

$$\frac{3}{2} \frac{P}{g} v_C \frac{dv_C}{dt} = P \left( \sin \alpha - \frac{k}{R} \cos \alpha \right) \frac{ds_C}{dt}.$$

The last multiplier is equal to  $v_C$ , and we finally obtain

$$a_c = \frac{2}{3} g \left( \sin \alpha - \frac{k}{R} \cos \alpha \right).$$

At  $k = 0$  this formula gives the result of the previous problem.

The frictional force can now be found from the equation  $Ma_c = P \sin \alpha - F$ , which does not change its form.

*Problem 41.* A uniform cylinder of weight  $P$  and radius  $r$  starts rolling from rest without slipping from a point on a cylindrical surface of radius  $R$  defined by angle  $\varphi_0$  (Fig.69).

Determine: 1) The pressure of the cylinder on the surface for any angle  $\varphi$ ; 2) The law of motion of the cylinder when angle  $\varphi_0$  is small. Neglect rolling friction.

*Solution.* 1) Acting on the cylinder in any position is force  $\mathbf{P}$ , the normal reaction  $\mathbf{N}$ , and the frictional force  $\mathbf{F}$ , without which rolling is impossible. The path of the centre  $C$  is known: a circle of radius  $R - r$ . To determine  $N$  we make use of equations (5.5). Directing the normal  $Cn$  inwards to the path and projecting all forces on this normal, we obtain

$$M \frac{v_C^2}{R-r} = N - P \cos \varphi.$$

The quantity  $v_C$  in this equation can be found from the theorem of the change in kinetic energy:

$$T - T_0 = \sum A_k^e.$$

In our case  $T_0 = 0$  and  $T = \frac{3}{4} M v_C^2$  (see Problem 29). Only force  $\mathbf{P}$  does any work, consequently,

$$\sum A_k^e = Ph = P(R - r)(\cos \varphi - \cos \varphi_0),$$

and equation takes the form

$$\frac{3}{4} M v_C^2 = P(R - r)(\cos \varphi - \cos \varphi_0).$$

Computing from here  $M v_C^2$ , we obtain finally

$$N = \frac{P}{3} (7 \cos \varphi - 4 \cos \varphi_0).$$

If, for example,  $\varphi_0 = 60^\circ$  and  $\varphi = 0^\circ$ , then  $N = \frac{5}{3} P$ .

2) To determine the law of motion of point  $C$ , differentiate equation with respect to time. We obtain

$$\frac{3}{2} \frac{P}{g} v_C \frac{dv_C}{dt} = -P(R - r) \sin \varphi \frac{d\varphi}{dt}.$$

In our case angle  $\varphi$  decreases when the cylinder moves, and  $\frac{d\varphi}{dt} < 0$ . Then

$$v_C = (R - r) \left| \frac{d\varphi}{dt} \right| = -(R - r) \frac{d\varphi}{dt}; \quad \frac{dv_C}{dt} = -(R - r) \frac{d^2\varphi}{dt^2}.$$

Substituting these expressions into the previous equation, we obtain finally

$$\frac{d^2\varphi}{dt^2} + \frac{2}{3} \frac{g}{R - r} \sin \varphi = 0.$$

If angle  $\varphi_0$  is small, then, as  $\varphi \leq \varphi_0$ , we can assume that  $\sin \varphi \approx \varphi$ , and the equation takes the form

$$\frac{d^2\varphi}{dt^2} + k^2 \varphi = 0,$$

where

$$k^2 = \frac{2}{3} \frac{g}{R - r}.$$

Consequently (see §3.1), the centre of the cylinder performs simple harmonic motion, its period being

$$T = \frac{2\pi}{k} = 2\pi \sqrt{\frac{3(R - r)}{2g}}.$$

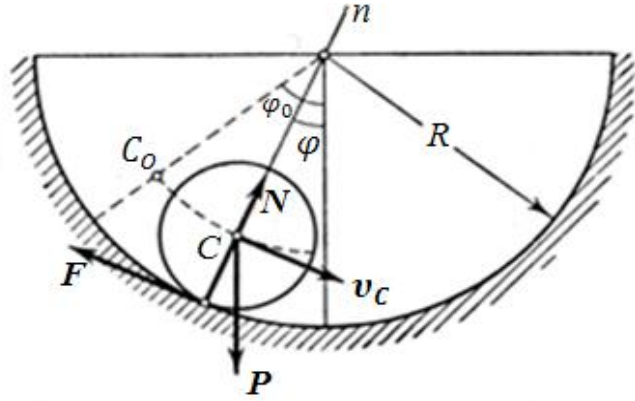


Fig. 69

**Problem 42.** The body in Fig. 70 rests at  $B$  on a piezoelectric sensor of an instrument for measuring pressure, and at  $A$  it is attached to a string  $AD$ .

When the system is in equilibrium,  $AC$  is horizontal and the pressure at  $B$  is  $Q = Q_0$ . Determine the moment of inertia  $J_C$  of the body with respect to an axis through the centre of mass  $C$ , if at the instant when the string is severed the pressure at  $B$  becomes  $Q_1$ , and the distance  $l$  is known.

**Solution.** 1) In the equilibrium position  $Q_0 l = P(l - a)$ , whence

$$a = \frac{P - Q_0}{P} l.$$

2) When the string is severed, the body begins plane motion. Its displacement in the initial time increment can be neglected. Then Eqs. (5.6) and (5.19), which are valid only for this initial time interval, will take the form

$$Ma_{Cx} = P - Q_1, \quad a_{Cy} = 0, \quad J_C \varepsilon = Q_1 a.$$

As  $a_{Cy} = 0$ , point  $C$  starts moving vertically down and point  $B$  slides horizontally (assuming the friction in the support to be very small). Erecting perpendiculars to the directions of these displacements, we find that the instantaneous centre of zero velocity is at point  $K$ . Consequently,  $v_C = \omega a$ . Assuming  $a = \text{const.}$  for the elementary time interval, we obtain, after differentiating,  $a_C = a\varepsilon$ . Then the first of equations gives

$$\frac{P}{g} a\varepsilon = P - Q_1.$$

Determining  $\varepsilon$  from here, we obtain finally

$$J_C = \frac{Q_1 a}{\varepsilon} = \frac{P - Q_1}{g} \frac{Q_1}{P - Q_1} a^2.$$

**Problem 43.** The weight of a motor car together with its wheels is  $P$ , the weight of each wheel is  $p$ , and their radii are  $r$  (Fig. 71). Acting on the rear (driving) wheels is a turning moment  $M_t$ . The car starts from rest and is subjected to the resistance of the air, which is proportional to the square of the translational velocity:  $R = \mu v^2$ . The

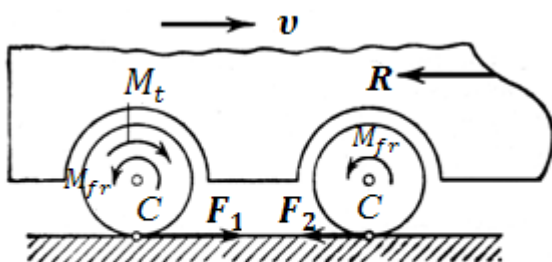


Fig. 71

frictional moment acting on the axle of each wheel is  $M_{fr}$ . Neglecting rolling friction, determine 1) the maximum velocity of the car; 2) the sliding friction acting on the driving and driven wheels during motion.

**Solution.** 1) To determine the maximum velocity, write the equation of motion from Eq. (5.38):

$$dT = dA^e + dA^i.$$

The kinetic energy of the car is equal to the energy of the body plus the energy of the wheels. Taking into account that  $P$  is the weight of the whole car and  $v_C = \omega r$ , and denoting the radius of gyration of each wheel by the symbol  $\rho$  we obtain

$$T = \frac{1}{2} \frac{P}{g} v_C^2 + 4 \left( \frac{1}{2} J_C \omega^2 \right) = \frac{1}{2g} \left( P + 4p \frac{\rho^2}{r^2} \right) v_C^2.$$

Of all the external forces, only the resistance of the air does work, as we have neglected rolling resistance, and in this case the work done by the frictional forces  $F_1$  and  $F_2$  of the wheels on the road is zero. Therefore,

$$dA^e = -\mu v_C^2 ds_C.$$

The work done by the internal forces (the torque and the friction in the axes) is

$$dA^i = (M_t - 4M_{fr}) d\varphi = (M_t - 4M_{fr}) \frac{ds_C}{r}.$$

Substituting all these expressions and dividing through by  $dt$  we obtain

$$\frac{1}{g} \left( p + 4p \frac{\rho^2}{r^2} \right) v_C \frac{dv_C}{dt} = \frac{1}{r} (M_t - 4M_{fr} - \mu r v_C^2) \frac{ds_C}{dt},$$

from which, cancelling out  $v_C = \frac{ds_C}{dt}$ , we find

$$\left( P + 4p \frac{\rho^2}{r^2} \right) a_C = \frac{g}{r} (M_t - 4M_{fr} - \mu r v_C^2).$$

When the velocity reaches its limiting value, the acceleration  $a_C$  becomes zero. Therefore  $v_C^{lim}$  can be found from the equation

$$M_t - 4M_{fr} - \mu r v_C^2 = 0,$$

whence

$$v_C^{lim} = \sqrt{\frac{M_t - 4M_{fr}}{\mu r}}.$$

2) To determine the frictional forces acting on each wheel, we deduce the equations of the rotation of the wheels about their axes. For the driving wheels, taking into account that the frictional force  $F_1$  acting on each of them is directed forward, we obtain

$$2 \frac{p}{g} \rho^2 \varepsilon = M_t - 2M_{fr} - 2F_1 r.$$

Since in rolling  $a_C = \varepsilon r$ , we obtain finally

$$F_1 = \frac{0.5M_t - M_{fr}}{r} - \frac{\rho^2 p}{r^2 g} a_C.$$

The frictional force  $F_2$  acting on each of the driven wheels is directed backwards. Therefore, for the driven wheels we have

$$\frac{p}{g} \rho^2 \varepsilon = F_2 r - M_{fr},$$

whence

$$F_2 = \frac{M_{fr}}{r} + \frac{\rho^2 p}{r^2 g} a_C.$$

## 6. THE PRINCIPLES OF DYNAMICS

### 6.1. D'Alembert's Principle for a Particle and a System

Consider a particle  $M$  moving along a given fixed curve or surface (Fig. 72). The resultant of all the active forces applied to the particle is denoted by the symbol  $\mathbf{F}^a$ . If the action of the constraint is replaced by its reaction  $\mathbf{N}$ , the particle can be considered as a free one moving under the action of forces  $\mathbf{F}^a$  and  $\mathbf{N}$ . Let us see what force  $\mathbf{F}^i$  should be added to the forces  $\mathbf{F}^a$  and  $\mathbf{N}$  to balance them. If the resultant of the forces  $\mathbf{F}^a$  and  $\mathbf{N}$  is  $\mathbf{R}$ , then, obviously, the required force  $\mathbf{F}^i = -\mathbf{R}$ .

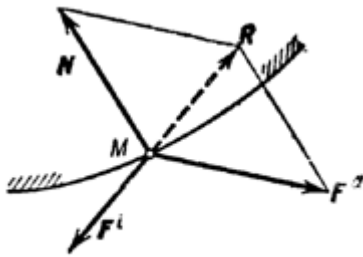


Fig. 72

Let us express force  $\mathbf{F}^i$  in terms of the acceleration of the moving particle. As, according to the fundamental law of dynamics,  $\mathbf{R} = m\mathbf{a}$ ,  $\mathbf{F}^i = -m\mathbf{a}$ .

The force  $\mathbf{F}^i$ , equal in magnitude to the product of the mass of the particle and its acceleration and directed oppositely to the acceleration, is called the *inertia force* of the particle.

Thus, if to the forces  $\mathbf{F}^a$  and  $\mathbf{N}$  is added the inertia force  $\mathbf{F}^i$ , the forces will be balanced, and we will have

$$\mathbf{F}^a + \mathbf{N} + \mathbf{F}^i = 0. \quad (6.1)$$

This equation states *D'Alembert's principle for a particle*: if at any given moment to the active forces and the reactions of the constraints acting on a particle is added the inertia force, the resultant force system will be in equilibrium and all the equations of statics will apply to it.

D'Alembert's principle provides a method of solving problems of dynamics by developing equations of motion in the form of equations of equilibrium.

In applying D'Alembert's principle it should always be remembered that actually only forces  $\mathbf{F}^a$  and  $\mathbf{N}$  are acting on a particle and that the particle is in motion. The inertia force does not act on a moving particle and the concept is introduced for the sole purpose of developing equations of dynamics with the help of the simpler methods of statics.

*D'Alembert's Principle for a System.* Consider a system of  $n$  particles. Let us select any particle of mass  $m_k$  and denote the resultants of all the external and internal forces applied to it by the symbols  $\mathbf{F}_k^{ext}$  and  $\mathbf{F}_k^{int}$ . If we add to these forces the inertia force  $\mathbf{F}_k^i = -m_k\mathbf{a}_k$ , then according to D'Alembert's principle for a single particle the force system  $\mathbf{F}_k^{ext}$ ,  $\mathbf{F}_k^{int}$ ,  $\mathbf{F}_k^i$  will be in equilibrium, and consequently,

$$\mathbf{F}_k^{ext} + \mathbf{F}_k^{int} + \mathbf{F}_k^i = 0.$$

Reasoning similarly for all the particles of the system, we arrive at the following result, which expresses D'Alembert's principle for a system: *if at any moment of time to the effective external and internal forces acting on every particle of a system are*

added the respective inertia forces, the resultant force system will be in equilibrium and all the equations of statics will apply to it.

We know from statics that the geometrical sum of balanced forces and the sum of their moments with respect to any centre  $O$  are zero; we know, further, from the principle of solidification, that this holds good not only for forces acting on a rigid body, but for any deformable system. Thus, according to D'Alembert's principle, we must have

$$\begin{aligned} \sum(\mathbf{F}_k^{ext} + \mathbf{F}_k^{int} + \mathbf{F}_k^i) &= 0, \\ \sum[\mathbf{m}_0(\mathbf{F}_k^{ext}) + \mathbf{m}_0(\mathbf{F}_k^{int}) + \mathbf{m}_0(\mathbf{F}_k^i)] &= 0. \end{aligned}$$

Let us introduce the following notation:

$$\mathbf{R}^i = \sum \mathbf{F}_k^i, \mathbf{M}_0^i = \sum \mathbf{m}_0(\mathbf{F}_k^i).$$

The quantities  $\mathbf{R}^i$  and  $\mathbf{M}_0^i$  are respectively the principal vector of the inertia forces and their principal moment with respect to a centre  $O$ . Taking into account that the sum of the internal forces and the sum of their moments are each zero we obtain

$$\sum \mathbf{F}_k^{ext} + \mathbf{R}^i = 0, \sum \mathbf{m}_0(\mathbf{F}_k^{ext}) + \mathbf{M}_0^i = 0. \quad (6.2)$$

Use of Eqs. (6.2), which follow from D'Alembert's principle, simplifies the process of problem solution because the equations do not contain the internal forces. Actually Eqs. (6.2) are equivalent to the equations expressing the theorems of the change in the momentum and the total angular momentum of a system, differing from them only in form.

## 6.2. The Principal Vector and the Principal Moment of the Inertia Forces of a Rigid Body

It follows from the Statics that a system of inertia forces applied to a rigid body can be replaced by a single force equal to  $\mathbf{R}^i$  and applied at the centre  $O$ , and a couple of moment  $\mathbf{M}_0^i$ . The principal vector of a system, it will be recalled, does not depend on the centre of reduction and can be computed at once. As  $\mathbf{F}_k^i = -m_k \mathbf{a}_k$ , then taking into account §5.3, we will have:

$$\mathbf{R}^i = -\sum m_k \mathbf{a}_k = -M \mathbf{a}_c. \quad (6.3)$$

Thus, the principal vector of the inertia forces of a moving body is equal to the product of the mass of the body and the acceleration of its centre of mass, and is opposite in direction to the acceleration.

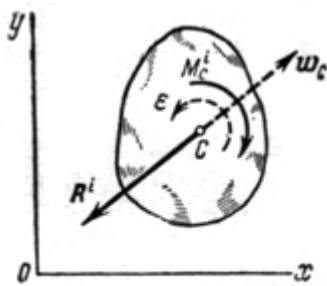


Fig. 73

Let us determine the principal moment of the inertia forces for particular types of motion.

1. *Translational Motion.* In this case a body has no rotation about its centre of mass  $C$ , from which we conclude that  $\sum \mathbf{m}_c(\mathbf{F}_k^{ext}) = 0$ , and Eq. (6.2) gives  $\mathbf{M}_C^i = 0$ .

Thus, in translational motion, the inertia forces of a rigid body can be reduced to a single resultant  $\mathbf{R}^i$  through the centre of mass of the body.

2. *Plane Motion.* Let a body have a plane of symmetry, and let it be moving parallel to the plane. By virtue of symmetry, the principal vector and the resultant couple of inertia forces lie, together with the centre of mass  $C$ , in that plane.

Therefore, placing the centre of reduction in point  $C$ , we obtain from Eq. (6.2)  $M_C^i = -\sum m_C(\mathbf{F}_k^{ext})$ . On the other hand (see § 5.9, 5.10),  $\sum m_C(\mathbf{F}_k^{ext}) = J_C\varepsilon$ . We conclude from this that

$$M_C^i = -J_C\varepsilon. \quad (6.4)$$

Thus, in such motion a system of inertia forces can be reduced to a resultant force  $\mathbf{R}^i$  (Eq. (6.3)) applied at the centre of mass  $C$  (Fig. 73) and a couple in the plane of symmetry of the body whose moment is given by Eq. (6.4). The minus sign shows that the moment  $M_C^i$  is in the opposite direction of the angular acceleration of the body.

3. *Rotation about an Axis through the Centre of Mass.* Let a body have a plane of symmetry, and let the axis of rotation  $Cz$  be normal to the plane through the centre of mass. This case will thus be a particular case of the previous motion. But here  $\mathbf{a}_c = 0$ , and consequently,  $\mathbf{R}^i = 0$ .

Thus, in this case a system of inertia forces can be reduced to a couple in the plane of symmetry of the body of moment

$$M_z^i = -J_C\varepsilon.$$

In applying Eqs. (6.3) and (6.4) to problem solutions, the magnitudes of the respective quantities are computed and the directions are shown in a diagram.

### 6.3. Virtual Displacements of a System. Degrees of Freedom

In determining the equilibrium conditions of a system by the methods of so-called graphical statics we had to consider the equilibrium of every body separately, replacing the action of all applied constraints by the unknown reaction forces. When the number of bodies in a system is large, this method becomes cumbersome, involving the solution of a large number of equations with many unknown quantities.

Now we shall make use of a number of kinematical and dynamical concepts to investigate a more general method for the solution of problems of statics, which makes it possible to determine at once, the equilibrium conditions for any mechanical system. The basic difference between this method and the methods of geometrical statics is that the action of constraints is taken into account not by introducing the

reaction forces but by investigating the possible displacements of a system if its equilibrium were disturbed. These displacements are known in mechanics by the name of *virtual displacements*.

Virtual displacements of the particles of a system must satisfy two conditions: 1) they must be infinitesimal, since if a displacement is finite the system will

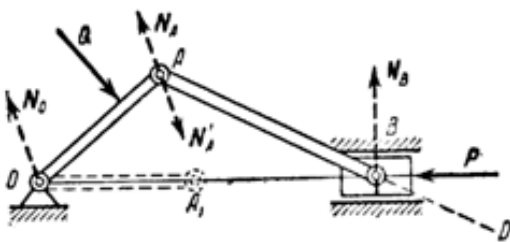


Fig. 74



occupy a new configuration in which the equilibrium conditions may be different; 2) they must be consistent with the constraints of the system, as otherwise we should change the character of the mechanical system under consideration. For instance, in the crankshaft mechanism in Fig. 74, a displacement of the points of the crank  $OA$  into configuration  $OA_1$  cannot be considered as a virtual displacement, as the equilibrium conditions under the action of forces  $P$  and  $Q$  will have changed. At the same time, even an infinitesimal displacement of point  $B$  of the connecting rod along  $BD$  would not be a virtual displacement: it would have been possible if the slides at  $B$  were replaced by a rocker, i.e., if it were a different mechanism.

Thus, we shall define as a virtual displacement of a system the sum total of any arbitrary infinitesimal displacements of the particles of the system consistent with all the constraints acting on the system at the given instant. We shall denote the virtual displacement of any point by an elementary vector  $\delta\mathbf{s}$  in the direction of the displacement.

In the most general case, the particles and bodies of a system may have a number of different virtual displacements (not considering  $\delta\mathbf{s}$  and  $-\delta\mathbf{s}$  as being different). For every system, however, depending on the type of constraints, we can specify a certain number of independent virtual displacements such that any other virtual displacements will be obtained as their geometrical sum. For example, a bead lying on a horizontal plane can move in many directions on the plane. Nevertheless, any virtual displacement  $\delta\mathbf{s}$  may be produced as the sum of two displacements  $\delta\mathbf{s}_1$  and  $\delta\mathbf{s}_2$  along two mutual perpendicular horizontal axes ( $\delta\mathbf{s} = \delta\mathbf{s}_1 + \delta\mathbf{s}_2$ ).

*The number of possible mutually independent displacements of a system is called the number of degrees of freedom of that system.* Thus, a bead on a plane (regarded as a particle) has two degrees of freedom. A crankshaft mechanism, evidently, has one degree of freedom. A free particle has three degrees of freedom (three independent displacements along mutually perpendicular axes). A free rigid body has six degrees of freedom (three translational displacements along orthogonal axes and three rotations about those axes).

*Ideal Constraints.* If a particle has for a constraint a smooth surface, the reaction  $\mathbf{N}$  of the constraint is normal to the surface and the elementary work done by the force  $\mathbf{N}$  in any virtual displacement of the particle is zero. It was shown that if we neglect rolling friction, the sum of the work done by the reaction forces  $\mathbf{N}$  and  $\mathbf{F}_{fr}$  in any virtual displacement of a rolling body is also zero. The internal forces of any non-deformable system also possess this property.

Let us introduce the following notation: the elementary work done by an active force  $\mathbf{F}^a$  in any virtual displacement  $\delta\mathbf{s}$  — the virtual work — shall be denoted by the symbol  $\delta A^a$  ( $\delta A^a = F^a \delta s \cos \alpha$ , where  $\alpha$  is the angle between the directions of the force and the displacement), and the virtual work done by the reaction  $\mathbf{N}$  of a constraint, by the symbol  $\delta A^N$ . Then for all the constraints considered here,

$$\sum \delta A_k^N = 0. \quad (6.5)$$

Constraints, in which the sum of the virtual work produced by all the reaction forces in any virtual displacement of a system is zero, are called *ideal constraints*.

We have seen that to such constraints belong all frictionless constraints along which a body slides and all rough constraints when a body rolls along them, neglecting rolling friction.

#### 6.4. The Principle of Virtual Work

Consider a system of material particles in equilibrium under the action of the applied forces and constraints, assuming all the constraints imposed on the system to be ideal. Let us take an arbitrary particle belonging to the system and denote the resultant of all the applied active forces (both external and internal) by the symbol  $\mathbf{F}_k^a$ , and the resultant of all the reactions of the constraints (also external and internal) by the symbol  $\mathbf{N}_k$ . Then, since this particle is in equilibrium together with the system,  $\mathbf{F}_k^a + \mathbf{N}_k = 0$  or  $\mathbf{N}_k = -\mathbf{F}_k^a$ .

Consequently, in any virtual displacement of the particle the virtual work  $\delta A_k^a$  and  $\delta A_k^N$  done by the forces  $\mathbf{F}_k^a$  and  $\mathbf{N}_k$  are equal in magnitude and opposite in sense and therefore vanish, i.e., we have:

$$\delta A_k^a + \delta A_k^N = 0.$$

Reasoning in the same way we obtain similar equations for all the particles of a system, adding which we obtain

$$\sum \delta A_k^a + \sum \delta A_k^N = 0.$$

But from the property of ideal constraints (6.5), the second summation is zero, whence

$$\sum \delta A_k^a = 0, \quad (6.6)$$

or

$$\sum (F_k^a \delta s_k \cos \alpha_k) = 0. \quad (6.7)$$

We have thus proved that if a mechanical system with ideal constraints is in equilibrium, the active forces applied to it satisfy the condition (6.6). The reverse is also true, i.e., if the active forces satisfy the condition (6.6), the system is in equilibrium. From this follows the principle of virtual work: *the necessary and sufficient conditions for the equilibrium of a system subjected to ideal constraints is that the total virtual work done by all the active forces is equal to zero for any and all virtual displacements consistent with the constraints.* Mathematically the necessary and sufficient condition for the equilibrium of any mechanical system is expressed by Eq. (6.6).

In analytical form this condition can be expressed as follows:

$$\sum (F_{kx}^a \delta x_k + F_{ky}^a \delta y_k + F_{kz}^a \delta z_k) = 0. \quad (6.8)$$

In Eq. (6.8)  $\delta x_k, \delta y_k, \delta z_k$  are the projections of the virtual displacements  $\delta \mathbf{s}_k$  of point on the coordinate axes. They are equal to the infinitesimal increments to the position coordinates of the point in its displacement and are computed in the same way as the differentials of coordinates.

The principle of virtual work provides in general form the equilibrium conditions of any mechanical system, whereas the methods of geometrical statics require the consideration of the equilibrium of every body of the system separately.

Furthermore, application of the principle of virtual work requires that only the active forces be considered and makes it possible to ignore all the unknown reactions of constraints, when the constraints are ideal.

### 6.5. The General Equation of Dynamics

The principle of virtual work gives a general method for solving problems of statics. On the other hand, D'Alembert's principle makes it possible to employ the methods of statics in solving dynamical problems. It seems obvious that by combining both these principles we can develop a general method for the solution of problems of dynamics.

Consider a system of material particles subjected to ideal constraints. If we add to all the particles subjected to active forces  $\mathbf{F}_k^a$  and the reaction forces  $\mathbf{N}_k$  the corresponding inertia forces  $\mathbf{F}_k^i = -m_k \mathbf{a}_k$ , then by D'Alembert's principle the resulting force system will be in equilibrium. If we now apply the principle of virtual work, we obtain

$$\sum \delta A_k^a + \sum \delta A_k^i + \sum \delta A_k^N = 0.$$

But from Eq. (6.5) the last summation is zero, and we finally obtain

$$\sum \delta A_k^a + \sum \delta A_k^i = 0. \quad (6.9)$$

Equation (6.9) represents the general equation of dynamics. It states that *in a moving system with ideal constraints the total virtual work done by all the active forces and all the inertia forces in any virtual displacement is zero at any instant.*

In analytical form Eq. (6.9) gives

$$\sum [(F_{kx}^a + F_{kx}^i) \delta x_k + (F_{ky}^a + F_{ky}^i) \delta y_k + (F_{kz}^a + F_{kz}^i) \delta z_k] = 0. \quad (6.10)$$

Equation (6.9) and (6.10) make it possible to develop the equations of motion for any mechanical system.

If a system consists of a number of rigid bodies, the relevant equations can be developed if to the active forces applied to each body are added a force equal to the principal vector of the inertia forces applied at any center, and a couple of moment equal to the principal moment of the inertia forces with respect to that center. Then the principle of virtual work can be used.

### 6.6. Solution of Problems

*Problem 44.* When a train accelerates, a load attached to a string hanging from the ceiling of a carriage deflects by an angle  $\alpha$  from the vertical (Fig. 75). Determine the acceleration of the carriage.

*Solution.* Acting on the load is the force of gravity  $\mathbf{P}$  and the reaction of the thread  $\mathbf{T}$ . Applying D'Alembert's principle, add to these

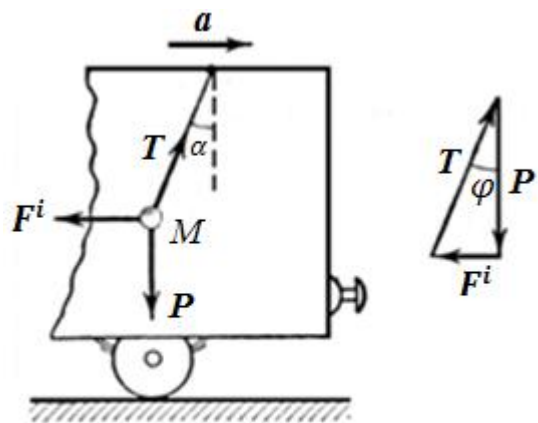


Fig. 75

forces the inertia force  $F^i$  directed opposite to the acceleration  $a$  of the carriage. In magnitude  $F^i = ma = \frac{P}{g}a$ . The forces  $P$ ,  $T$  and  $F^i$  are balanced. Constructing a closed force triangle and taking into account that  $\varphi = \alpha$ , we find

$$F^i = P \tan \alpha \text{ or } \frac{P}{g}a = P \tan \alpha.$$

Hence, the acceleration of the carriage is  $a = g \tan \alpha$ .

**Problem 45.** Neglecting the mass of all the rotating parts of the centrifugal-type governor in Fig. 76 as compared with the mass of the balls  $B$  and  $D$ , determine the angle  $\alpha$  defining the position of relative equilibrium of rod  $AB$  of length  $l$  if the governor rotates with a constant angular velocity  $\omega$ .

**Solution.** In order to determine the position of relative equilibrium (with respect to a set of axes rotating together with the governor) add, according to Eq. (6.1), to the force of gravity  $P$  and the reaction  $N$  acting on ball  $B$  the transport inertia force  $F_{tr}^i$ . As  $\omega$  const,  $a_{tr} = a_{tr}^n = BC\omega^2 = l\omega^2 \sin \alpha$ , whence  $F_{tr}^i = ml\omega^2 \sin \alpha$ . Writing the equilibrium equation for the projections on axis  $B\tau$ , which is perpendicular to  $AB$ , we have

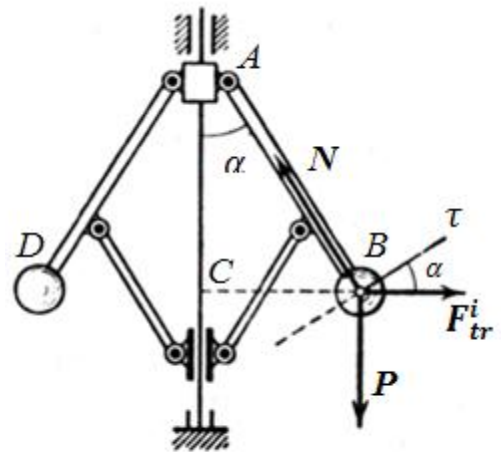


Fig. 76

$$-P \sin \alpha + F_{tr}^i \cos \alpha = 0.$$

Hence, substituting  $F_{tr}^i$  for its expression and eliminating  $\sin \alpha$  (not considering the solution for  $\alpha = 0$ ), we obtain

$$-g + l\omega^2 \cos \alpha = 0,$$

whence

$$\cos \alpha = \frac{g}{l\omega^2}.$$

As  $\cos \alpha \leq 1$ , equilibrium at  $\alpha \neq 0$  is possible only when  $\omega^2 > \frac{g}{l}$ .

**Problem 46.** The semicircle  $BCD$  of radius  $R$  in Fig. 77 rotates about a vertical axis with a constant angular velocity  $\omega$ . A ring  $M$  starts slipping along it without friction from a point  $B$  slightly off the axis of rotation. Determine the relative velocity  $v_1$  of the ring at point  $C$  if its initial velocity  $v_0 = 0$ .

**Solution.** The velocity  $v_1$ , can be determined from the theorem of the change in kinetic energy. In order to write Eq. (5.36), which expresses the theorem, compute the work done by forces  $P$  and  $F_{tr}^i$ , where  $F_{tr}^i = m\omega^2 x$  (the

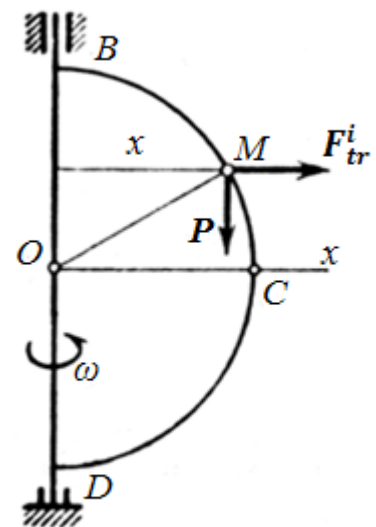


Fig. 77

work done by the reaction  $\mathbf{N}$  is zero). Assuming approximately  $x_B = 0$ , we obtain

$$A_{(BC)}(\mathbf{F}_{tr}^i) = \int_{(B)}^{(C)} F_{tr\ x}^i dx = m\omega^2 \int_0^R x dx = \frac{1}{2}m\omega^2 R^2.$$

Furthermore,  $A_{(BC)}(\mathbf{P}) = PR$ . Substituting these expressions into Eq. (5.36) and taking into account that  $v_0 = 0$ , we have

$$\frac{mv_1^2}{2} = mR \left( g + \frac{1}{2}\omega^2 R \right),$$

whence

$$v_1 = \sqrt{2gR \left( 1 + \frac{\omega^2 R}{2g} \right)}.$$

*Problem 47.* Two weights  $P_1$  and  $P_2$  are connected by a thread and move along a horizontal plane under the action of a force  $\mathbf{Q}$  applied to the first weight (Fig. 78a). The coefficient of friction of the weights on the plane is  $f$ . Determine the accelerations of the weights and the tension in the thread.

*Solution.* Denote all the external forces acting on the system and add to them the inertia forces of the weights. As both weights are translated with the same acceleration  $\mathbf{a}$ , then in magnitude

$$F_1^i = \frac{P_1}{g}a \text{ and } F_2^i = \frac{P_2}{g}a.$$

The forces are directed as shown. Then frictional forces are

$$F_1 = fP_1, F_2 = fP_2.$$

According to D'Alembert's principle, the force system must be in equilibrium. Writing the equilibrium equation in terms of the projections on horizontal axis, we find

$$Q - f(P_1 + P_2) - \frac{1}{g}(P_1 + P_2)a = 0,$$

whence

$$a = \left( \frac{Q}{P_1 + P_2} - f \right) g.$$

Evidently, the weights will move if  $f < \frac{Q}{P_1 + P_2}$ .

In our force system the required tension in the thread is an internal force. To determine it we divide the system and apply D'Alembert's principle to one of the weight, say the second (Fig. 78b). Acting on it is force  $\mathbf{P}_2$ , the normal reaction  $\mathbf{N}_2$ , the frictional force  $\mathbf{F}_2$ , and tension  $\mathbf{T}$  in the thread. Add to them the inertia force  $\mathbf{F}_2^i$  and write the equilibrium in terms of the projection on horizontal axis. We have

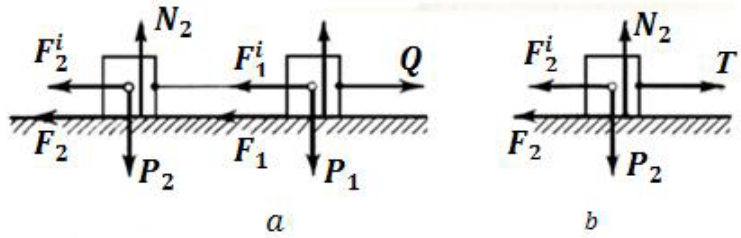


Fig. 78

$$T - fP_2 - \frac{P_2}{g}a = 0.$$

Substituting the earlier found value of  $a$ , we obtain finally

$$T = \frac{QP_2}{P_1 + P_2}.$$

It is interesting to note that the tension in the thread does not depend on the friction and, given the same total weight of the system, it decreases with the reduction of the second (rear) weight. That is why, for example, in making up a goods train it is better to place the heavier vans closer to the locomotive.

*Problem 48.* Solve problem 22 with the help of D'Alembert's principle and also determine the tension in the thread.

*Solution.* 1) Considering the drum and the load as a single system, we add to the bodies of the system inertia forces (Fig. 79). Load  $A$  is in translational motion and for it  $R^i = \frac{Q}{g}a_A = \frac{Q}{g}r\varepsilon$ . The inertia forces of the drum can be reduced to a couple of moment  $M_0^i$  equal in magnitude to  $J_0\varepsilon = \frac{P}{g}\rho^2\varepsilon$  and directed opposite the rotation. Writing now for all the forces the equilibrium conditions in the form  $\sum m_0(\mathbf{F}_k) = 0$ , we obtain

$$|M_0^i| + R^i r - Qr = 0,$$

or

$$\frac{P}{g}\rho^2\varepsilon + \frac{Q}{g}r^2\varepsilon - Qr = 0,$$

from which we find

$$\varepsilon = \frac{Qgr}{P\rho^2 + Qr^2}.$$

2) Considering now load  $A$  separately and adding to the active forces  $\mathbf{Q}$  and  $\mathbf{T}$  the inertia force  $\mathbf{R}^i$ , we obtain from the equilibrium conditions that the tension in the thread

$$T = Q - R^i = Q \left( 1 - \frac{r\varepsilon}{g} \right) = \frac{PQ\rho^2}{P\rho^2 + Qr^2}.$$

*Problem 49.* Determine the forces acting on a spinning flywheel, assuming its mass to be distributed along the rim. The weight of the flywheel is  $P$ , its radius  $r$  and its angular velocity  $\omega$ .

*Solution.* The required force is an internal one. In order to determine it, cut the rim into two and apply D'Alembert's principle to one portion (Fig. 80). We denote the action of the separated half by two equal forces  $\mathbf{F}'$  equal in magnitude to the required force  $\mathbf{F}$ . For each element of the rim, the inertia force (a centrifugal force) is directed along the radius. These concurrent forces intersecting at  $O$  have a resultant equal to the principal vector  $\mathbf{R}^i$  of the inertia forces directed, by virtue of symmetry, along

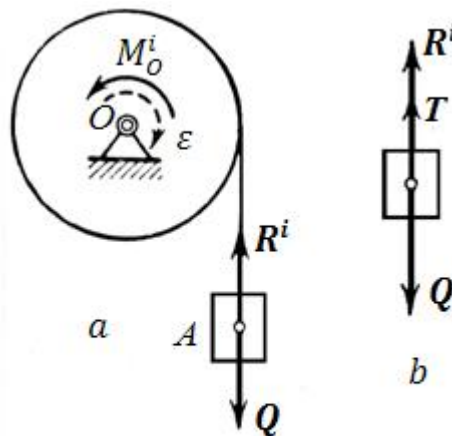


Fig. 79

axis  $Ox$ . By Eq. (6.3),  $R^i = ma_c = mx_c\omega^2$ , where  $m$  is the mass of half the rim and  $x_c$  is the coordinate of the mass centre of the semicircular arc, which is equal to  $\frac{2r}{\pi}$ . Therefore,

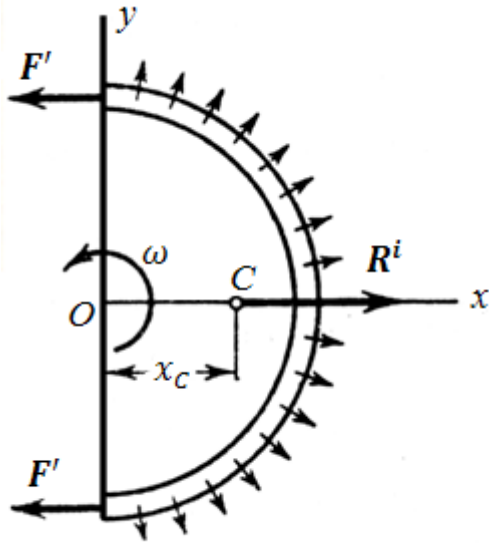


Fig. 80

$$m = \frac{P}{2g}, R^i = \frac{Pr\omega^2}{\pi g}.$$

The equilibrium conditions give  $2F = R^i$ , and finally

$$F = \frac{Pr\omega^2}{2\pi g}.$$

This formula can be used to determine the limiting angular velocity beyond which a flywheel made of a specific material may be torn apart.

**Problem 50.** A homogeneous rod  $AB$  of length  $l$  and weight  $P$  is hinged at  $A$  to a vertical shaft rotating with an angular velocity  $\omega$  (Fig. 81). Determine the tension  $T$  in the horizontal thread securing the rod at an angle  $\alpha$  to the shaft.

**Solution.** Applying D'Alembert's principle, we add to the external forces  $P, T, X_A$  and  $Y_A$  acting on the rod the inertia forces. For each element of the rod of mass  $\Delta m$  the centrifugal inertia force is  $\Delta m\omega^2 x$ , where  $x$  is the distance of the element from the axis of rotation  $Ay$ . The resultant of these parallel forces distributed according to a linear law passes through the centre of gravity of triangle  $ABE$ , i.e., at a distance  $h = \frac{2}{3}l \cos \alpha$  from the  $Ax$  axis. As this resultant is equal to the principal vector of the inertia forces, then, by Eq. (6.3),

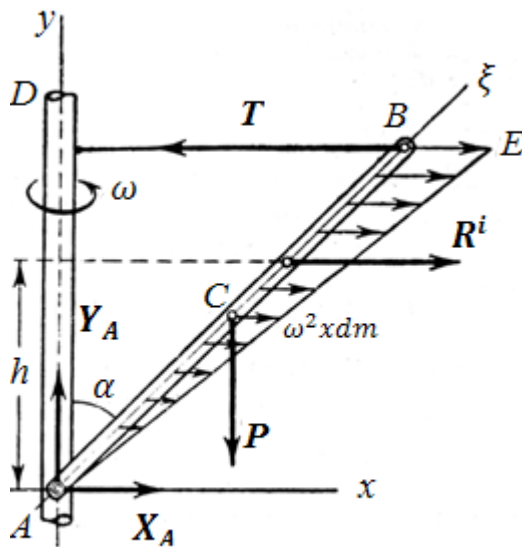


Fig. 81

$R^i = ma_c = m\omega^2 x_c = \frac{P}{g}\omega^2 \frac{1}{2} \sin \alpha$

(here  $x_c$  is the coordinate of the centre of gravity of the rod).  
Writing now the statics equation  $\sum m_A(\mathbf{F}_k) = 0$ , we obtain

$$Tl \cos \alpha - R^i h - P \frac{l}{2} \sin \alpha = 0.$$

Substituting the values of  $R^i$  and  $h$  into this equation, we obtain finally

$$T = P \left( \frac{l\omega^2}{3g} \sin \alpha + \frac{1}{2} \tan \alpha \right).$$

*Problem 51.* A homogeneous rod  $AB$  of mass  $m$  and length  $l$  rotates about an axis perpendicular to it with an angular velocity  $\omega$  and an angular acceleration  $\varepsilon$  (Fig. 82). Determine the stresses generated by the rotation in a cross section of the rod at a distance  $x$  from the axis.

*Solution.* The required forces are internal. To determine them, cut the rod into

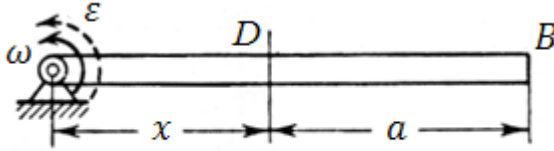


Fig. 82

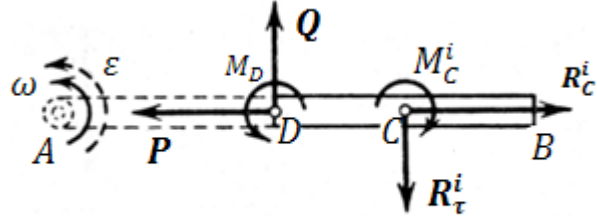


Fig. 83

two and consider the motion of portion  $DB$  of length  $a = l - x$  (Fig. 83). The action of the removed portion  $AD$  is replaced by a force applied at the centre  $D$  of the cross section, which we shall represent by its components  $\mathbf{P}$  and  $\mathbf{Q}$ , and a couple of moment  $M_D$ . The quantities  $\mathbf{P}$ ,  $\mathbf{Q}$  and  $M_D$  will specify the required stresses in section  $D$  of the rod, i.e., the forces with which portions  $AD$  and  $DB$  act on one another. To compute these quantities we use D'Alembert's principle. First let us find the principal vector  $\mathbf{R}^i$  of the inertia forces of portion  $DB$ , and their principal moment  $M_C^i$  with respect to the centre of mass  $C$  of the portion. The mass of portion  $DB$  and the coordinate  $x_C = AC$  of its centre of mass are

$$m_1 = \frac{l-x}{l}m, \quad x_C = \frac{l+x}{2}.$$

Then, from Eqs. (6.3), we find magnitude of vectors  $\mathbf{R}_\tau^i$  and  $\mathbf{R}_C^i$ :

$$|\mathbf{R}_\tau^i| = m_1 |a_{c\tau}| = m_1 x_C |\varepsilon| = m |\varepsilon| \frac{l^2 - x^2}{2l},$$

$$R_C^i = m_1 a_{cn} = m_1 x_C \omega^2 = m \omega^2 \frac{l^2 - x^2}{2l}.$$

Furthermore, from Eq. (6.4),  $M_C^i = -J_C \varepsilon$ . As in this case  $J_C = \frac{ma^2}{12}$ , then

$$|M_C^i| = m \frac{(l-x)^3}{12l} |\varepsilon|.$$

The forces  $\mathbf{R}_\tau^i$  and  $\mathbf{R}_C^i$  and the moment  $M_C^i$  are directed as shown in the Fig.83.

Thus, all the inertia forces of portion  $DB$  of the rod are replaced by a force  $\mathbf{R}^i$  applied at  $C$ , whose components are  $\mathbf{R}_\tau^i$  and  $\mathbf{R}_C^i$  and a couple of moment  $M_C^i$ . Writing now the equilibrium conditions  $\sum F_{kx} = 0$ ,  $\sum F_{ky} = 0$ ,  $\sum m_D (F_k) = 0$  for the active forces and the inertia forces applied to portion  $DB$ , we obtain

$$R_C^i - P = 0; \quad Q - |R_\tau^i| = 0; \quad M_D - |M_C^i| - |R_\tau^i| \frac{a}{2} = 0.$$

From this we find finally that acting at section  $D$  of the rod are 1) a tensile force  $P = R_C^i$  and 2) a transverse force  $Q = |R_\tau^i|$ , and 3) a couple with a bending moment

$$M_D = |M_C^i| + |R_\tau^i| \frac{a}{2} = \frac{m|\varepsilon|}{12l} [(l-x)^3 + 3(l+x)(l-x)^2],$$

or finally



$$M_D = \frac{m|\varepsilon|}{6l} (l-x)^2 (2l+x).$$

The value of forces  $P, Q$  and the bending moment will be greatest at the cross section  $x = 0$ .

**Problem 52.** Two equal bars of length  $l$  and weight  $p$  each are welded at right angles to a vertical shaft of length  $b$  at distance  $h$  from each other (Fig. 84). Determine the dynamical pressures acting on the shaft if it rotates with a constant angular velocity  $\omega$ .

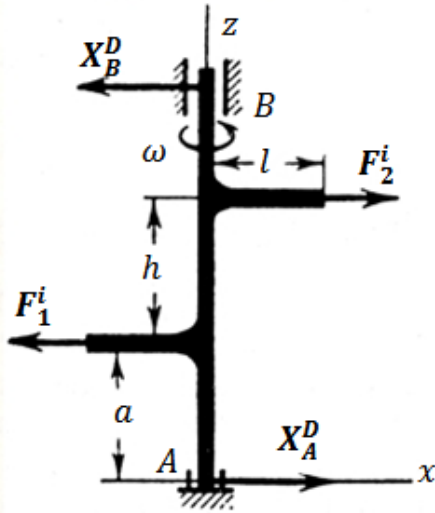


Fig. 84

**Solution.** The centrifugal inertia forces in each rod are equal in magnitude:

$$F_1^i = F_2^i = \frac{P}{g} \frac{1}{2} \omega^2,$$

and they make a couple which, apparently, is balanced by the couple  $X_A^D, X_B^D$ . The moments of these couples are equal in magnitude. Consequently,  $X_A^D b = F_1^i h$ , whence

$$X_A^D = X_B^D = \frac{F_1^i h}{b} = \frac{plh}{2gb} \omega^2.$$

The couple is continuously in the  $Axz$  plane, which rotates with the body.

**Problem 53.** Find the relation between the moment  $M$  of the couple acting on the crankshaft mechanism in Fig. 85 and the pressure  $P$  on the piston when the system is in equilibrium. The crank is of length  $OA = r$  and the connecting rod is of length  $AB = l$ .

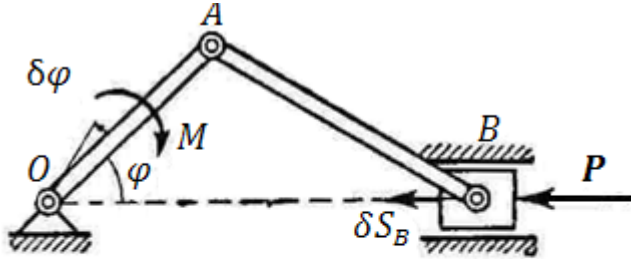


Fig. 85

**Solution.** Equilibrium conditions (6.6) give

$$M\delta\varphi - P\delta s_B = 0 \quad \text{or} \quad M\omega_{OA} =$$

$$Pv_B,$$

since  $\delta\varphi = \omega_{OA} dt$  and  $\delta s_B = v_B dt$ . The relation between  $v_B$  and

$\omega_{OA}$  can be found by the methods of kinematics:

$$v_B = \omega_{OA} r \left( 1 + \frac{r \cos \varphi}{\sqrt{l^2 - r^2 \sin^2 \varphi}} \right) \sin \varphi.$$

Referring to this result, we find

$$M = Pr \left( 1 + \frac{r \cos \varphi}{\sqrt{l^2 - r^2 \sin^2 \varphi}} \right) \sin \varphi.$$

*Problem 54.* For the reduction gear (fig. 86), find the relation between the torque  $M_A$  applied to the driving shaft  $A$  and the resistance moment  $M_B$  applied to the driven shaft  $B$  when both shafts are rotating uniformly.

The number of teeth in the gears is:  $z_1 = 120, z_2 = 40, z_3 = 30, z_4 = 50$ .

*Solution.* The relation between  $M_A$  and  $M_B$  will be the same in uniform rotation as in equilibrium. Therefore, from Eq. (6.6) we have

$M_A \delta \varphi_A - M_B \delta \varphi_B = 0$  or  $M_A \omega_A = M_B \omega_B$ ,  
as  $\delta \varphi_A = \omega_A dt$ , and  $\delta \varphi_B = \omega_B dt$ . Hence, referring to the kinematic methods, we find

$$\frac{\omega_B}{\omega_A} = 1 + \frac{z_1 z_3}{z_2 z_4}, \quad M_A = \frac{\omega_B}{\omega_A} M_B = \left(1 + \frac{z_1 z_3}{z_2 z_4}\right) M_B = 2.8 M_B.$$

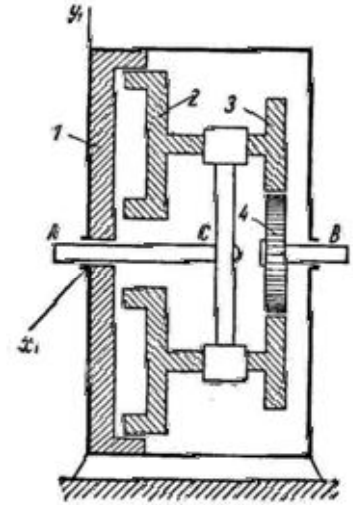


Fig. 86

*Problem 55.* Find the relation between forces  $P$  and  $Q$  in the hoisting mechanism in Fig. 87, whose parts are housed in the box  $K$ , if it is known that in one revolution of the crank handle  $AB = l$  the screw  $D$  moves out by  $h$ .

*Solution.* From Eq. (6.6) we have

$$Pl \delta \varphi_{AB} - Q \delta s_D = 0.$$

Assuming that when the handle is rotated uniformly the screw also moves up uniformly, we have

$$\frac{\delta \varphi_{AB}}{2\pi} = \frac{\delta s_D}{h} \text{ or } \delta \varphi_{AB} = \frac{2\pi}{h} \delta s_D.$$

Substituting this expression for  $\delta \varphi_{AB}$  into the foregoing equation, we obtain

$$Q = \frac{2\pi l}{h} P.$$

Note that this simple problem could just not be solved by the methods of geometrical statics as the parts of the mechanism are unknown.

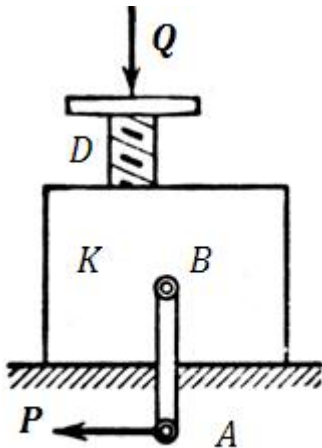


Fig. 87

*Problem 56.* Two beams are hinged together at  $C$  and loaded as shown in Fig. 88a. Neglecting the weight of the beams, determine the pressure on support  $B$ .

*Solution.* Replace the support at  $B$  by a force  $N_B$ , which is equal in magnitude to the required pressure (Fig. 88b). For a virtual displacement of the system Eq. (6.6) gives

$$N_B \delta s_B - P \delta s_E = 0.$$

The relation between  $\delta s_B$  and  $\delta s_E$  is found from the proportions

$$\frac{\delta s_B}{a} = \frac{\delta s_C}{l_1}; \quad \frac{\delta s_E}{b} = \frac{\delta s_C}{l_2},$$

whence

$$\delta s_E = \frac{bl_1}{al_2} \delta s_B,$$

and consequently

$$N_B = \frac{bl_1}{al_2} P.$$

If we used the methods of geometrical statics we would have to consider the equilibrium of each beam separately, introduce the reactions of the other supports, and then eliminate them from the obtained set of equilibrium equations.

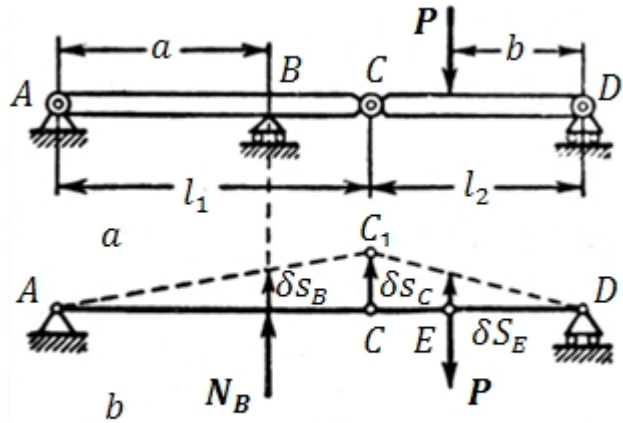


Fig. 88

**Problem 57.** The epicycles gear train in Fig. 89 consists of a gear 1 of radius  $r_1$ , an arm  $AB$  mounted on axle  $A$  independently of the gear, and a gear 2 of radius  $r_2$  mounted on the arm at  $B$  as shown. Acting on the arm is a torque  $M$ , and acting on the gears are resistance moments  $M_1$  and  $M_2$ , respectively. Determine the values of  $M_1$  and  $M_2$  at which the mechanism is in equilibrium.

**Solution.** The mechanism has two degrees of freedom, since it has two possible independent displacements: the rotation of the arm  $AB$  when gear 1 is at rest, and the rotation of gear 1 when the arm is at rest. First consider a virtual displacement of the system in which gear 1 remains at rest (Fig. 89a). For this displacement Eq. (6.6) gives

$$M\delta\varphi_{AB} - M_2\delta\varphi_2 = 0.$$

But when gear 1 is at rest the contact point of the gears will be the instantaneous centre of zero velocity of gear 2, and consequently  $v_B = \omega_2 r_2$ . At the same time,

$$v_B = \omega_{AB}(r_1 + r_2).$$

Hence  $\omega_2 r_2 = \omega_{AB}(r_1 + r_2)$  or  $\delta\varphi_2 r_2 = \delta\varphi_{AB}(r_1 + r_2)$ , and we obtain

$$M_2 = \frac{r_2}{r_1 + r_2} M.$$

Now consider a virtual displacement in which the arm  $AB$  remains at rest (Fig. 89b). For this displacement Eq. (6.6) gives

$$M_1\delta\varphi_1 - M_2\delta\varphi_2 = 0.$$

But when the arm is at rest

$$\frac{\delta\varphi_2}{\delta\varphi_1} = \frac{\omega_2}{\omega_1} = \frac{r_1}{r_2} \text{ and } M_1 = \frac{r_1}{r_2} M_2.$$

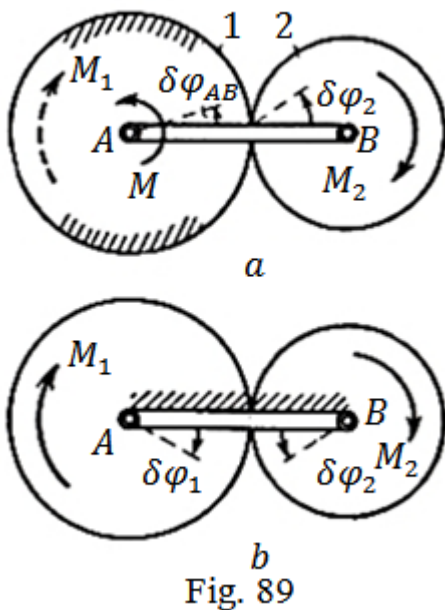


Fig. 89

We finally obtain

$$M_1 = \frac{r_1}{r_1 + r_2} M, \quad M_2 = \frac{r_2}{r_1 + r_2} M.$$

**Problem 58.** Determine the relation between forces  $Q$  and  $P$  at which the press in Fig. 90 is in equilibrium if angles  $\alpha$  and  $\beta$  are known. Neglect the weight of the rods.

**Solution.** To give an example of the analytical method of solution, let us take equilibrium condition (6.8). Placing the origin of a coordinate system in the fixed point  $A$  and drawing the  $x$  and  $y$  axes as shown, we obtain

$$Q_{1x}\delta x_1 + Q_{2x}\delta x_2 + P_{3y}\delta y_3 = 0,$$

since all the other projections of the forces vanish.

To find  $\delta x_1, \delta x_2, \delta y_3$  compute the coordinates  $x_1, x_2, y_3$  of the points of application of the forces, expressing them in terms of the angles  $\alpha$  and  $\beta$ . Denoting the length of each rod by  $a$ , we obtain

$$x_1 = a \cos \alpha, \quad x_2 = a \cos \alpha + 2a \cos \beta, \quad y_3 = a(\sin \beta + \sin \alpha),$$

differentiating which, we find  $\delta x_1 = -a \sin \alpha \delta \alpha$ ,  $\delta x_2 = -a(\sin \alpha \delta \alpha + 2 \sin \beta \delta \beta)$ ,  $\delta y_3 = a(\cos \beta \delta \beta + \cos \alpha \delta \alpha)$ .

Substituting these expressions and taking into account that  $Q_{1x} = Q$ ,  $Q_{2x} = -Q$ , and  $P_{3y} = -P$ , we have

$$2Q \sin \beta \delta \beta - P(\cos \beta \delta \beta + \cos \alpha \delta \alpha) = 0.$$

To find the relation between  $\delta \alpha$  and  $\delta \beta$  we make use of the fact that  $AB = \text{const}$ . Therefore,  $2a(\cos \alpha + \cos \beta) = \text{const}$ . Differentiating this equation, we obtain

$$\sin \alpha \delta \alpha + \sin \beta \delta \beta = 0 \quad \text{and} \quad \delta \alpha = -\frac{\sin \beta}{\sin \alpha} \delta \beta.$$

Substituting the expression for  $\delta \alpha$ , we have

$$2Q \sin \beta - P(\cos \beta - \cot \alpha \sin \beta) = 0,$$

whence

$$P = \frac{2Q}{\cot \beta - \cot \alpha}.$$

At an angle  $\beta$  very close to  $\alpha$  the pressure  $P$  will be very large.

**Problem 59.** A centrifugal-type governor consists of two balls  $A_1$  and  $A_2$  of weight  $P$  each (Fig. 91). The slide  $C_1C_2$  weighs  $Q$ , the governor rotates about the vertical axis with a uniform angular velocity  $\omega$ . Neglecting the weight of the rods, determine angle  $\alpha$ , if  $OA_1 = OA_2 = l$ , and  $OB_1 = OB_2 = B_1C_1 = B_2C_2 = a$ .

**Solution.** Adding to the active forces  $P_1, P_2$  and  $Q_3$  the centrifugal inertia forces  $F_1^i$  and  $F_2^i$  (the inertia force of the slide will, evidently, be zero), we write the general equation of dynamics in the form (6.10). Computing the projections of all the forces on the coordinate axes, we have

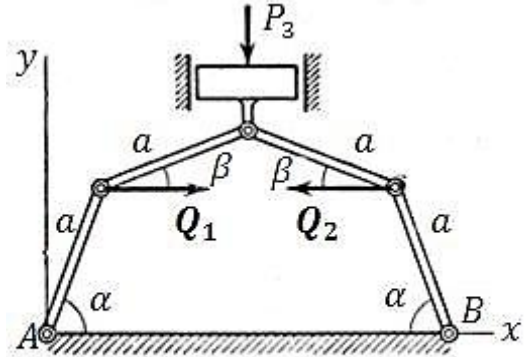


Fig. 90

$$P_1 \delta x_1 + P_2 \delta x_2 - F_1^i \delta y_1 + F_2^i \delta y_2 + Q_3 \delta x_3 = 0.$$

We also have

$$Q_3 = Q; \quad P_1 = P_2 = P; \quad F_1^i = F_2^i = \\ = \frac{P}{g} a_A = \frac{P}{g} \omega^2 l \sin \varphi.$$

The coordinates of the points of application of the forces are

$$x_1 = x_2 = l \cos \varphi, \quad y_2 = -y_1 = l \sin \varphi, \quad x_3 = \\ = 2a \cos \varphi.$$

Differentiating these expressions, we find

$$\delta x_1 = \delta x_2 = -l \sin \varphi \delta \varphi; \quad \delta y_2 = -\delta y_1 \\ = l \cos \varphi \delta \varphi; \quad \delta x_3 = -2a \sin \varphi \delta \varphi.$$

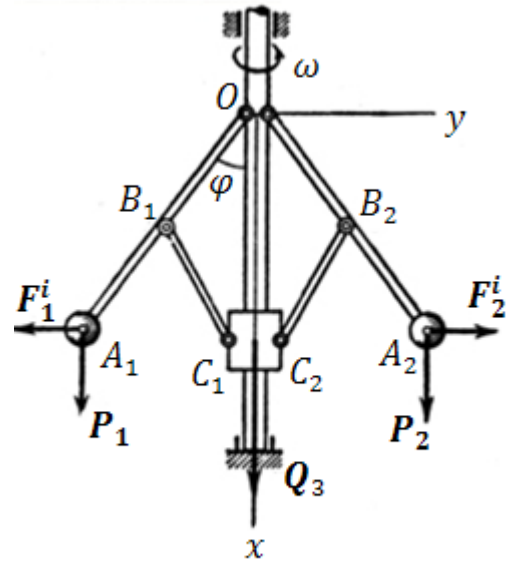


Fig. 91

Substituting all these expressions, we obtain

$$\left( -2Pl \sin \varphi + 2 \frac{P}{g} l^2 \omega^2 \sin \varphi \cos \varphi - 2Qa \sin \varphi \right) \delta \varphi = 0,$$

whence we finally have

$$\cos \alpha = \frac{Pl + Qa}{Pl^2 \omega^2} g.$$

As  $\cos \varphi \leq 1$ , the balls will move apart when

$$\omega^2 > \frac{Pl + Qa}{Pl^2} g.$$

Angle  $\varphi$  increases with  $\omega$  and tends to  $90^\circ$  when  $\omega \rightarrow \infty$ .

*Problem 60.* In the hoist mechanism in Fig. 92, a torque  $M$  is applied to gear 2 of weight  $P_2$  and radius of gyration  $\varrho_2$ . Determine, the acceleration of the lifted load  $A$  of weight  $Q$ , neglecting the weight of the string and the friction in the axles. The drum on which the string winds and the gear 1 attached to it have a total weight  $P_1$  and a radius of gyration  $\varrho_1$ . The radii of the gears are  $r_1$  and  $r_2$ , and of the drum  $r$ .

*Solution.* Draw the active force  $Q$  and torque  $M$  (forces  $P_1$  and  $P_2$  do no work) and add to them the inertia force  $F_A^i$  of the load and the couples of moments  $M_1^i$  and  $M_2^i$  to which the inertia forces of the rotating bodies are reduced. In magnitude these quantities are

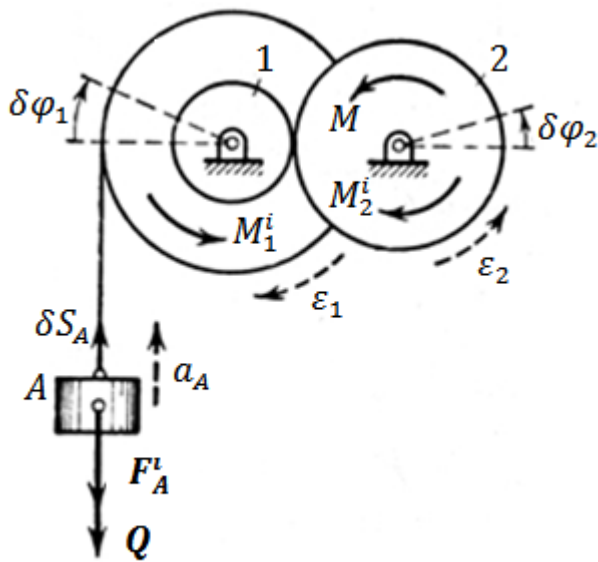


Fig. 92

Finally the equation of motion takes the form

$$Q \left( 1 + \frac{a_A}{g} \right) r + \frac{P_1}{g} \varrho_1^2 \varepsilon_1 + \frac{P_2}{g} \varrho_2^2 \varepsilon_2 \frac{r_1}{r_2} - M \frac{r_1}{r_2} = 0.$$

Now express the quantities  $\varepsilon_1$  and  $\varepsilon_2$  in terms of the required acceleration  $a_A$ . Taking into account that  $\varepsilon_1$  and  $\varepsilon_2$  are related in the same way as  $\omega_1$  and  $\omega_2$ , we obtain

$$\varepsilon_1 = \frac{a_A}{r}, \quad \varepsilon_2 = \frac{r_1}{r_2} \varepsilon_1 = \frac{r_1 a_A}{r_2 r}.$$

And finally we have

$$a_A = \frac{\frac{r_1}{r_2} M - rQ}{rQ + \frac{\varrho_1^2}{r} P_1 + \frac{\varrho_2^2 r_1^2}{r r_2^2} P_2} g.$$

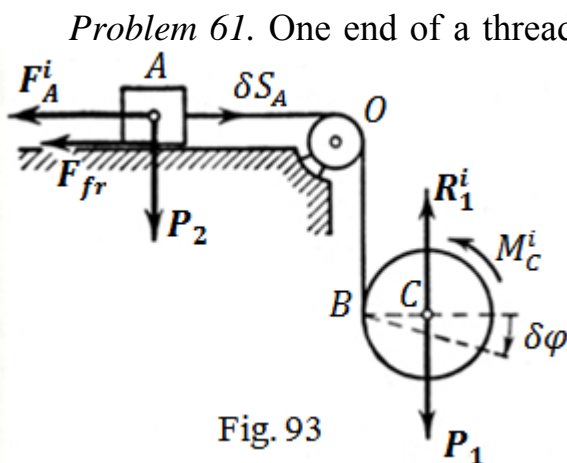


Fig. 93

**Problem 61.** One end of a thread is wound on a uniform cylinder of weight  $P_1$  (Fig. 93). The thread passes over a pulley  $O$ , and its other end is attached to a load  $A$  of weight  $P_2$  which slides on a horizontal plane, the coefficient of friction being  $f$ . Neglecting the mass of the pulley, and the string, determine the acceleration of the load and of the centre  $C$  of the cylinder.

*Solution.* If motion starts from rest, the centre of the cylinder  $C$  will move vertically, and the system has two degrees of freedom

(the rotation of the cylinder with respect to the thread when the load is at rest and the displacement of the load when the cylinder does not rotate).

$$F_A^i = \frac{Q}{g} a_A, \quad |M_1^i| = \frac{P_1}{g} \varrho_1^2 \varepsilon_1, \\ |M_2^i| = \frac{P_2}{g} \varrho_2^2 \varepsilon_2.$$

The directions of all the vectors are shown in the Fig 92. Eq. (6.9) for a virtual displacement of the system, we obtain

$$-(Q + F_A^i) \delta S_A - M_1^i \delta \varphi_1 + (M - M_2^i) \delta \varphi_2 = 0.$$

Expressing all the displacements in terms of  $\delta \varphi_1$ , we have

$$\delta S_A = r \delta \varphi_1, \quad \frac{\delta \varphi_1}{\delta \varphi_2} = \frac{\omega_1}{\omega_2} = \frac{r_2}{r_1} \\ \text{and } \delta \varphi_2 = \frac{r_1}{r_2} \delta \varphi_1.$$

Add to the acting forces  $\mathbf{P}_1$ ,  $\mathbf{P}_2$ , and  $\mathbf{F}_{fr}$  the inertia forces of the cylinder reduced to a principal vector  $\mathbf{R}_1^i$  and a couple of moment  $M_C^i$ , and the inertia force  $\mathbf{F}_A^i$  of the load. In magnitude

$$F_A^i = \frac{P_2}{g} a_A, R_1^i = \frac{P_1}{g} a_C, |M_C^i| = J_C \varepsilon = \frac{P_1}{2g} r^2 \frac{a_C - a_A}{r}.$$

The last equality follows from the fact that, if point  $C$  of the cylinder has a velocity  $v_C$ , and point  $B$  (together with the string) a velocity  $v_B = v_A$ , then the angular velocity of the cylinder  $\omega = \frac{v_C - v_A}{r}$ , and, consequently,  $\varepsilon = \frac{a_C - a_A}{r}$ . Furthermore, for the cylinder  $J_C = 0.5mr^2$ , where  $r$  is the radius of the cylinder.

Now consider a virtual displacement  $\delta s_A$  of the system in which the cylinder does not rotate and is translated together with the load. The couple of moment  $M_C^i$  does no work in this displacement and from Eq. (6.9) we obtain

$$(-F_{fr} - F_A^i - R_1^i + P_1) \delta s_A = 0,$$

whence, as  $F_{fr} = fP_2$ , we find

$$\frac{P_1}{g} a_C + \frac{P_2}{g} a_A = P_1 - fP_2.$$

Consider the other independent virtual displacement in which the load  $A$  remains at rest while the cylinder turns about point  $B$  (which in this displacement is the instantaneous centre of rotation) through angle  $\delta\varphi$ . For this displacement, Eq. (6.9) gives

$$(P_1 - R_1^i)r\delta\varphi - M_C^i\delta\varphi = 0$$

Substituting the expressions for  $R_1^i$  and  $M_C^i$  we obtain

$$3a_C - a_A = 2g.$$

Solving equations simultaneously, we obtain the required accelerations

$$a_A = \frac{P_1 - 3fP_2}{P_1 + 3P_2} g, a_C = \frac{P_1 + (2 - f)P_2}{P_1 + 3P_2} g.$$

## 7. LAGRANGIAN DYNAMICS

### 7.1. Generalized Coordinates, Velocities and Accelerations

As the problems in dynamics become more complex it, naturally, becomes increasingly difficult to work out the solutions. This difficulty is associated not only with the solution of the equations of motion, but with their formulation as well. In fact, the derivation of the basic differential equations of motion in a form suitable for a particular complicated problem may well be the most difficult part of the investigation. A number of methods, more powerful than those hitherto considered in this manual, have been developed for deriving the equations for these more involved situations. Perhaps the most generally useful of these more advanced methods for engineering problems is that of Lagrange, who has put the basic equations of motion in such a form that the simplifying features of a particular problem can be utilized most advantageously. In the present chapter we shall derive Lagrange's equations.

One of the principal advantages of Lagrange's method is that one uses for each problem that coordinate system which most conveniently describes the motion. We have already seen that the position of a particle can be described in a large number of different ways, and we have found in the problems already discussed that the choice of a proper coordinate system may introduce a considerable simplification into the solution of a problem. In general, the requirement for a system of coordinates is that the specification of the coordinates must locate completely the position of each part of the system. This means that there must be one coordinate associated with each degree of freedom of the system. More exactly, there must be at least one coordinate associated with each degree of freedom. So called *non-holonomic* systems exist, for which, because of the particular geometrical constraints involved, more coordinates are required than there are degrees of freedom. Such systems are not often encountered and will not be considered here. We shall restrict the following treatment to systems whose coordinates are independent, in the sense that a change can be given to any one of the coordinates without changing any of the other coordinates. By the *generalized coordinates*  $(q_1, q_2, \dots, q_s)$  we shall mean a set of independent coordinates, equal in number to the  $s$  degrees of freedom of the system. We use the word "generalized" to emphasize the fact that such coordinates are not necessarily of the type of the simple  $(x, y, z)$  or  $(r, \theta, \varphi)$  systems and to indicate that they are not necessarily lengths or angles, but may be any quantity appropriate to the description of the position of the system.

The  $(x_k, y_k, z_k)$  coordinates of a point  $k$  are expressible in terms of the generalized coordinates  $(q_1, q_2, \dots, q_s)$  by functional relations:

$$\begin{aligned}x_k &= x_k(q_1, q_2, \dots, q_s), \\y_k &= y_k(q_1, q_2, \dots, q_s), \\z_k &= z_k(q_1, q_2, \dots, q_s).\end{aligned}\tag{7.1}$$

For example, if  $(q_1, q_2, q_3)$  are the cylindrical coordinates of a point  $(r, \theta, \varphi)$ ,



the foregoing equations become:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z.$$

We have supposed that the relation between the coordinate systems does not involve time. In the more general treatment in which  $x = x(q_1, q_2, \dots, q_s, t)$  analysis can proceed along essentially the same lines. The equations of motion in generalized coordinates for any particular system could always be obtained by writing the equations first in an  $(x, y, z)$  system, and then transforming to the  $q$ 's by Eqs.(7.1). This procedure usually leads to involved algebraic manipulations, and it is better to make the transformation in general terms and to write the equations of motion directly in generalized coordinates.

Thus, the parameters of any dimensions  $(q_1, q_2, \dots, q_s)$  describing the configuration of the system in space are called *generalized coordinates*. Their first derivatives with respect to time  $(\dot{q}_1, \dot{q}_2, \dots, \dot{q}_s)$  are called *generalized velocities* and the second derivatives  $(\ddot{q}_1, \ddot{q}_2, \dots, \ddot{q}_s)$  are said to be *generalized accelerations*.

## 7.2. Generalized Forces

Let the  $q_i$ , for  $i = 1, s$ , be a set of generalized coordinates which uniquely specifies the instantaneous position of some dynamical system which has  $s$  degrees of freedom. Here, it is assumed that each of  $q_i$  can vary independently. Since the generalized coordinates are independent, their elemental increments  $(\delta q_1, \delta q_2, \dots, \delta q_s)$  are also independent. Each of these quantities defines a virtual displacement of the system. Let the system be under the action of the active forces  $\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_n$ . Since the radius-vector of any point is a function of their coordinates  $\mathbf{r}_k = x_k \mathbf{i} + y_k \mathbf{j} + z_k \mathbf{k}$ , one can write

$$\mathbf{r}_k = \mathbf{r}_k(q_1, q_2, \dots, q_s). \quad (7.2)$$

We now calculate the virtual (elementary) work in terms of displacements of the  $n$  particles assumed to make up the system and the forces  $\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_n$  acting on them. The virtual work is

$$\sum \delta A_k^a = \sum_{k=1}^n \mathbf{F}_k \cdot \delta \mathbf{r}_k. \quad (7.3)$$

Now, since  $\mathbf{r}_k = x_k \mathbf{i} + y_k \mathbf{j} + z_k \mathbf{k}$ , we can write:

$$\delta \mathbf{r}_k = \sum_{i=1}^s \frac{\partial \mathbf{r}_k}{\partial q_i} \delta q_i. \quad (7.4)$$

for  $k=1, n$ .

Substituting (7.4) into (7.3), one can obtain

$$\sum \delta A_k^a = \sum_{k=1}^n \mathbf{F}_k \sum_{i=1}^s \frac{\partial \mathbf{r}_k}{\partial q_i} \delta q_i.$$

The above expression can be rearranged to give

$$\sum \delta A_k^a = \sum_{i=1}^s Q_i \cdot \delta q_i, \quad (7.5)$$

where

$$Q_i = \sum_{k=1}^n \mathbf{F}_k \frac{\partial \mathbf{r}_k}{\partial q_i}. \quad (7.6)$$

Here the  $Q_i$  are called *generalized forces*. Note that generalized forces do not necessarily have the dimensions of force. However, the product  $Q_i \delta q_i$  must have the

dimension of work. Thus, if particular  $q_i$  is a lineal parameter, then the associated  $Q_i$  is a force. Conversely, if  $q_i$  is an angle, then  $Q_i$  is a torque.

Formula (7.5) represents the elementary work of the acting forces in terms of generalized coordinates. This definition of the  $Q_i$  indicates the way in which they can be determined in specific problems. To find  $Q_i$ , the total work done by all of external forces during an infinitesimal displacement  $\delta q_i$  of one of the coordinates is calculated, and  $Q_i$  is then obtained by dividing this total work by  $\delta q_i$ .

Thus, the generalized forces are coefficients of the increments of the generalized coordinates in the expression for the total elementary work of all forces applied to the system. It is obvious that the number of generalized forces is equal to the number of degrees of freedom.

If the forces acting on the system are conservative, their total elementary work can be written  $\delta A = -\delta\Pi$ , where  $\Pi$  is a potential energy of the system which is a function of the coordinates  $x_k, y_k, z_k$ . But these coordinates are the functions of generalized coordinates. Thus,  $\Pi = \Pi(q_1, q_2, \dots, q_s)$ . Calculating a total differential of this function, one can obtain

$$\delta A = -\delta\Pi = -\left[\frac{\partial\Pi}{\partial q_1}\delta q_1 + \frac{\partial\Pi}{\partial q_2}\delta q_2 + \dots + \frac{\partial\Pi}{\partial q_s}\delta q_s\right].$$

Comparing this expression with equation (7.5) we have

$$Q_1 = -\frac{\partial\Pi}{\partial q_1}, Q_2 = -\frac{\partial\Pi}{\partial q_2}, \dots, Q_s = -\frac{\partial\Pi}{\partial q_s}. \quad (7.7)$$

Therefore, when the forces applied to the system are conservative, generalized forces are the partial derivatives of the potential energy of the system with respect to correspondent generalized coordinates taken with sign minus.

### 7.3. Conditions of Equilibrium in Terms of Generalized Coordinates

In accordance with principle of virtual works the necessary and sufficient conditions for the equilibrium of a system subjected to ideal constraints is that the total virtual work done by all the active forces is equal to zero for any and all virtual displacements consistent with the constraints, so  $\sum_{k=1}^n \delta A_k = 0$ . In terms of generalized coordinates this condition, taking into account equation (7.5), can be written

$$\sum_{i=1}^s Q_i \cdot \delta q_i = 0.$$

Since  $\delta q_i$ , as independent variables, can not be equal to 0, the generalized forces  $Q_i$  must disappear in an equilibrium position, i.e.,  $Q_i = 0, i=1,2,\dots,s$ .

Upon solving the above equations with respect to  $s$  unknown generalized coordinates  $q_i$ , one may always obtain all possible system's equilibrium positions.

Hence, *a holonomic system with perfect constraints is in its equilibrium only if all generalized forces corresponding to generalized coordinates are equal to zero.*

For conservative systems, taking into account Eq.(7.7), we have

$$\frac{\partial\Pi}{\partial q_1} = 0, \frac{\partial\Pi}{\partial q_2} = 0, \frac{\partial\Pi}{\partial q_s} = 0.$$

## 7.4. Lagrange's Equations of Motion

Let us consider the general equation of dynamics:

$$\sum \delta A_k^a + \sum \delta A_k^{in} = 0, \quad (7.8)$$

where  $\delta A_k^a$  and  $\delta A_k^{in}$  are virtual works of applied impressed forces and inertial forces respectively.

By analogy with Eq.(7.5) one can write

$$\sum \delta A_k^{in} = \sum_{i=1}^s Q_i^{in} \cdot \delta q_i,$$

where  $Q_i^{in}$  are generalized forces of inertia which can be defined as

$$Q_i^{in} = \sum_{k=1}^n \mathbf{F}_k^{in} \frac{\partial \mathbf{r}_k}{\partial q_i}. \quad (7.9)$$

Then from Eq.(7.8) we have for  $i=1, \dots, s$

$$Q_i + Q_i^{in} = 0. \quad (7.10)$$

Let us express  $Q_i^{in}$  in terms of kinetic energy  $T$ . Since  $\mathbf{F}_k^{in} = -m_k \mathbf{a}_k = -m_k \frac{d\mathbf{v}_k}{dt}$ , Eq.(7.9) yields

$$-Q_i^{in} = \sum_{k=1}^n m_k \frac{d\mathbf{v}_k}{dt} \frac{\partial \mathbf{r}_k}{\partial q_i}. \quad (7.11)$$

Notice that

$$\frac{d\mathbf{v}_k}{dt} \frac{\partial \mathbf{r}_k}{\partial q_i} = \frac{d}{dt} \left( \mathbf{v}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_i} \right) - \mathbf{v}_k \frac{d}{dt} \left( \frac{\partial \mathbf{r}_k}{\partial q_i} \right). \quad (7.12)$$

Further,

$$\frac{d}{dt} \left( \frac{\partial \mathbf{r}_k}{\partial q_i} \right) = \frac{\partial}{\partial q_i} \left( \frac{d\mathbf{r}_k}{dt} \right) = \frac{\partial \mathbf{v}_k}{\partial q_i}. \quad (7.13)$$

Any position vector associated with holonomic system has the form of Eq.(7.2). Since the generalized coordinates are themselves functions of time, the first derivative of the position vector with respect to time is

$$\dot{\mathbf{r}}_k = \sum_{i=1}^s \frac{\partial \mathbf{r}_k}{\partial q_i} \dot{q}_i, \quad (7.14)$$

where  $\dot{\mathbf{r}}_k = \frac{d\mathbf{r}_k}{dt}$ ,  $\dot{q}_i = \frac{dq_i}{dt}$ .

Since all position vectors do not depend on  $\dot{q}_i$ , the partial derivative  $\frac{\partial \mathbf{r}_k}{\partial q_i}$  do not depend on  $\dot{q}_i$  either. Hence,  $\frac{\partial}{\partial \dot{q}_i} \left( \frac{\partial \mathbf{r}_k}{\partial q_i} \right) = 0$ .

Therefore, differentiation of Eq. (7.14) with respect to  $\dot{q}_i$  yields

$$\frac{\partial \mathbf{r}_k}{\partial q_i} = \frac{\partial \dot{\mathbf{r}}_k}{\partial \dot{q}_i} = \frac{\partial \mathbf{v}_k}{\partial \dot{q}_i}. \quad (7.15)$$

Making use of formulas (7.13) and (7.15), expression (7.12) can be written

$$\frac{d\mathbf{v}_k}{dt} \cdot \frac{\partial \mathbf{r}_k}{\partial q_i} = \frac{d}{dt} \left( \mathbf{v}_k \cdot \frac{\partial \mathbf{v}_k}{\partial \dot{q}_i} \right) - \mathbf{v}_k \cdot \frac{\partial \mathbf{v}_k}{\partial q_i} = \frac{d}{dt} \left( \frac{1}{2} \frac{\partial v_k^2}{\partial \dot{q}_i} \right) - \frac{1}{2} \frac{\partial v_k^2}{\partial q_i}.$$

Taking into account that mass is constant, the Eq. (7.11) yields

$$-Q_i^{in} = \frac{d}{dt} \left[ \frac{\partial}{\partial \dot{q}_i} \left( \sum_{k=1}^n \frac{m_k v_k^2}{2} \right) \right] - \frac{\partial}{\partial q_i} \left( \sum_{k=1}^n \frac{m_k v_k^2}{2} \right) = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i}, \quad (7.16)$$

where  $T = \sum_{k=1}^n \frac{m_k v_k^2}{2}$  is kinetic energy of a system.

Hence, from Eq.(7.10) we have finally

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = Q_i, \quad (7.17)$$

where  $i=1, \dots, s$ .

Eqs. (7.17) are called *Lagrange's equations*. They represent the differential equations of motion of a system in terms of generalized coordinates. It is obvious that their number is equal to the number of degrees of freedom of a system.

If all impressed forces acting on a system are conservative, one can obtain taking into account formulas (7.7)

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial \Pi}{\partial q_i} = 0 \text{ or } \frac{d}{dt} \left( \frac{\partial (T-\Pi)}{\partial \dot{q}_i} \right) - \frac{\partial (T-\Pi)}{\partial q_i} = 0. \quad (7.18)$$

The last equation is valid since a potential energy  $\Pi$  depends only on generalized coordinates and does not depend on generalized velocities. Hence,

$$\frac{\partial \Pi}{\partial \dot{q}_i} = 0.$$

It is helpful to introduce a function  $L$ , called the *Lagrangian*, which is defined as the difference between the kinetic and potential energies of the dynamical system under investigation:  $L=T-\Pi$ . Then, from (7.18), we have

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0. \quad (7.19)$$

## 7.5. Solution of Problems

*Problem 62.* Resolve problem 61 by means of Lagrange's equations.

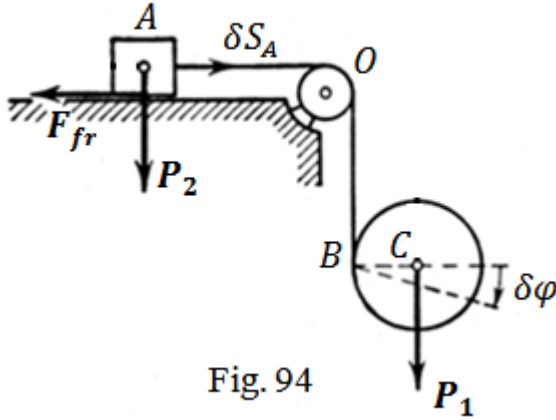


Fig. 94

*Solution.* Mechanical system has two degrees of freedom (the rotation of the cylinder with respect to the thread, when the load  $A$  is at rest, and the displacement of the load when the cylinder does not rotate, fig. 94).

Let us choose as generalized coordinates displacement of bodies with thread  $S_A$  and angle of rotation of cylinder  $\varphi$ . Then we have two independent virtual displacements  $\delta S_A$  and  $\delta \varphi$ . Now consider a virtual

displacement  $\delta S_A$  of the system in which the cylinder does not rotate and is translated together with the load. The force  $P_2$  does no work in this displacement. There are forces  $P_1$  and  $F_{fr}$  which produce work on elemental displacement  $\delta S_A$ . It equals

$$\delta A = (-F_{fr} + P_1) \delta S_A,$$

whence, as  $F_{fr} = P_2 f$ , we find generalized force corresponding to generalized coordinate  $S_A$ ,

$$Q_{S_A} = P_1 - f P_2.$$

Consider the other independent virtual displacement in which the load  $A$  remains at rest while the cylinder turns about point  $B$  (which in this displacement is the instantaneous centre of rotation) through angle  $\delta \varphi$ . For this displacement there is

only one force of the weight of the cylinder which produces work on elemental displacement  $\delta\varphi$ . It equals

$$\delta A = P_1 \delta s_C = P_1 r \delta\varphi,$$

where  $r$  is the radius of the cylinder. So we have found generalized force corresponding to generalized coordinate  $\varphi$ ,

$$Q_\varphi = P_1 r.$$

Now determine kinetic energy of a system. It equals  $T = T_1 + T_2$ , where  $T_1$  is a kinetic energy of a cylinder and  $T_2$  is kinetic energy of a load.

Cylinder is in a resultant motion which consists of the transport motion (this is a motion of a tread with velocity  $v_A$ ) and relative motion (this is a motion of a cylinder with respect to the thread, i.e., plane motion). Therefore,

$$T_1 = \frac{P_1 v_C^2}{2g} + \frac{I_C \omega^2}{2},$$

where  $v_C$  is absolute velocity of the center  $C$ ,  $J_C$  is a moment of inertia of cylinder and  $\omega$  is its relative angular velocity since transport motion is a translational one).

From the problem 61 we have  $J_C = 0,5m_1 r^2$ . Calculate  $v_C$ :

$$v_C = v_A + r\omega.$$

Taking into account that  $v_A = \dot{s}_A$  and  $\omega = \dot{\varphi}$ , we have

$$T_1 = \frac{P_1 (\dot{s}_A + r\dot{\varphi})^2}{2g} + \frac{P_1 r^2 \dot{\varphi}^2}{4g}.$$

Load  $A$  is in translational motion, then

$$T_2 = \frac{P_2 v_A^2}{2g} = \frac{P_2 \dot{s}_A^2}{2g}.$$

Thus,

$$T = \frac{P_1 (\dot{s}_A + r\dot{\varphi})^2}{2g} + \frac{P_1 r^2 \dot{\varphi}^2}{4g} + \frac{P_2 \dot{s}_A^2}{2g}.$$

Motion of the system considered is governed by Lagrange's equations of the following form:

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{s}_A} \right) - \frac{\partial T}{\partial s_A} &= Q_{s_A}, \\ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\varphi}} \right) - \frac{\partial T}{\partial \varphi} &= Q_\varphi. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial T}{\partial \dot{s}_A} &= \frac{P_1 (\dot{s}_A + r\dot{\varphi})}{g} + \frac{P_2 \dot{s}_A}{g}, & \frac{\partial T}{\partial \dot{\varphi}} &= \frac{P_1 (\dot{s}_A + r\dot{\varphi})r}{g} + \frac{P_1 r^2 \dot{\varphi}}{2g}, \\ \frac{\partial T}{\partial s_A} &= \frac{\partial T}{\partial \varphi} = 0. \end{aligned}$$

Hence, we have following Lagrange's equations:

$$\begin{aligned} P_1 (\ddot{s}_A + r\ddot{\varphi}) + P_2 \ddot{s}_A &= g(P_1 - fP_2), \\ 2\ddot{s}_A + 3r\ddot{\varphi} &= 2g. \end{aligned}$$

But  $\ddot{s}_A$  is acceleration of the load  $A$ , i.e.,  $a_A$ . At the same time it is a transport acceleration of the cylinder. The product  $r\ddot{\varphi} = r\varepsilon$  represents relative acceleration of

the point C. Since relative and transport accelerations have the same sense,  $\ddot{s}_A + r\ddot{\phi} = a_C$ . Then we obtain

$$\begin{aligned} P_1 a_C + P_2 a_A &= g(P_1 - fP_2), \\ 3a_C - a_A &= 2g. \end{aligned}$$

This set of equations gives

$$a_A = \frac{P_1 - 3fP_2}{P_1 + 3P_2} g, \quad a_C = \frac{P_1 + (2-f)P_2}{P_1 + 3P_2} g.$$

**Problem 63.** An uniform and thin bar 2 of mass  $m$  and length  $l$  is hinged to link 1 which rotates with a constant angular speed  $\omega$  (Fig. 95). Derive the differential equation of motion of link 2 by means of Lagrange's equations. Neglect the mass of the link 1.

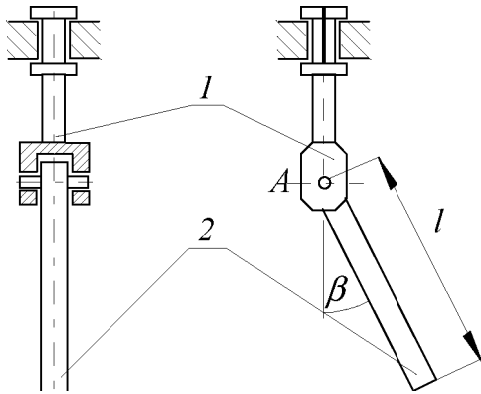


Fig. 95

**Solution.** Assume that  $\beta$  is the generalized coordinate. Absolute angular velocity of the link 2 is a sum of the transport angular velocity of the link 1  $\omega$  and the relative velocity of the link 2 with respect to the link 1 (Fig. 96).

$$\omega_2 = \omega + i_2 \dot{\beta} = k_1 \omega + i_2 \dot{\beta}.$$

Since

$$k_1 = j_2 \sin \beta + k_2 \cos \beta,$$

the absolute angular velocity of the link 2 is

$$\omega_2 = i_2 \dot{\beta} + j_2 \omega \sin \beta + k_2 \omega \cos \beta.$$

Its components are

$$\omega_{2x} = \dot{\beta}, \quad \omega_{2y} = \omega \sin \beta, \quad \omega_{2z} = \omega \cos \beta.$$

The link 2 performs rotational motion about point O. The moment of inertia of the link 2 about z axis is zero. Hence, its total kinetic energy is

$$T = \frac{1}{2} I (\omega_{2x}^2 + \omega_{2y}^2) = \frac{1}{2} I (\dot{\beta}^2 + \omega^2 \sin^2 \beta).$$

There is only one force of the weight of the link 2 which produces work on elemental displacement  $\delta\beta$ . It equals  $\delta A = -\frac{1}{2} mgl \sin \beta \delta\beta$ . Hence, generalized force is  $Q_\beta = -\frac{l}{2} mgl \sin \beta$ .

Motion of the system considered is governed by Lagrange's equations of the following form:

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\beta}} \right) - \frac{\partial T}{\partial \beta} = Q_\beta,$$

where

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\beta}} \right) = I \ddot{\beta}, \quad \frac{\partial T}{\partial \beta} = I \omega^2 \sin \beta \cos \beta.$$

Then one can obtain

$$I \ddot{\beta} - I \omega^2 \sin \beta \cos \beta + \frac{1}{2} mgl \sin \beta = 0.$$

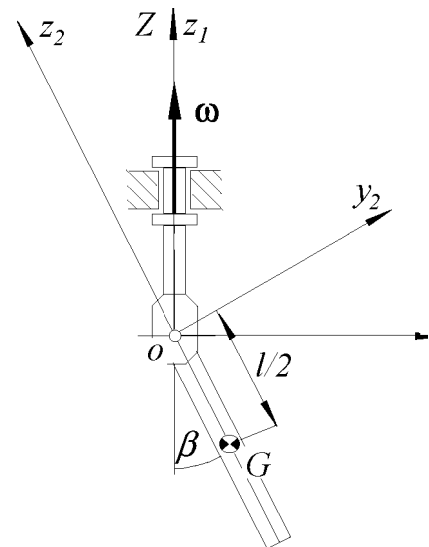


Fig. 96

Since  $I = \frac{1}{3}ml^2$  the final form of equation of motion is

$$\ddot{\beta} - \omega^2 \sin \beta \cos \beta + \frac{3}{2l}g \sin \beta = 0.$$

**Problem 64.** The bead 1 which can be considered as a particle of mass  $m$ , may slides without any friction along the slide 2. The slide 2 rotates with the constant angular velocity  $\omega$  about the vertical axis  $Z$  (Fig. 97). By means of Lagrange's equations derive equation of motion of the bead 1 and determine all possible equilibrium positions.

Given are:  $R = 25 \text{ cm}$ ,  $\omega = 10 \text{ rad/s}$ .

**Solution.** The angle  $\alpha$  can be considered as the generalized coordinate.

In Fig. 98 the inertial system of coordinates is denoted by  $XYZ$ . System of coordinates  $xyz$  is rigidly attached to the slide and rotates with the angular velocity  $\omega$  about

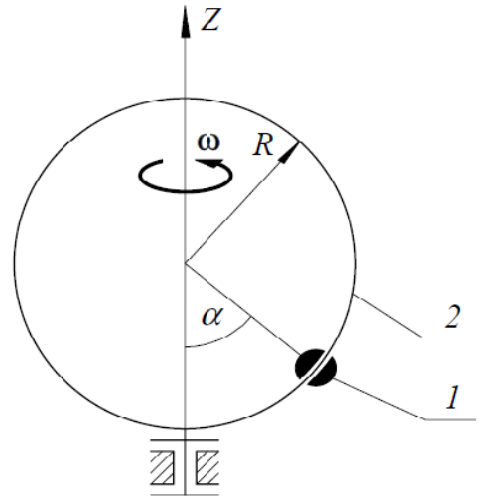


Fig. 97

axis  $Z$ . This is a transport angular velocity of the bead. The relative velocity of the bead is its velocity along slide. Vector of the absolute velocity of the bead 1 is

$$\mathbf{v}_a = \mathbf{v}_r + \mathbf{v}_{tr},$$

where  $v_r = R\dot{\alpha}$ ,  $v_{tr} = R\omega \sin \alpha$ . Since  $\mathbf{v}_r \perp \mathbf{v}_{tr}$ ,  $v_a^2 = \dot{\alpha}^2 R^2 + \omega^2 R^2 \sin^2 \alpha$ .

Hence,

$$T = \frac{m v_a^2}{2} = \frac{1}{2} m (\dot{\alpha}^2 R^2 + \omega^2 R^2 \sin^2 \alpha).$$

By analogy with problem 63  $Q_\alpha = -mgR \sin \alpha$ . Lagrange's equations may be taken in the following form:

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\alpha}} \right) - \frac{\partial T}{\partial \alpha} = Q_\alpha,$$

where

$$\frac{\partial T}{\partial \dot{\alpha}} = mR^2 \dot{\alpha}, \quad \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\alpha}} \right) = mR^2 \ddot{\alpha},$$

$$\frac{\partial T}{\partial \alpha} = mR^2 \omega^2 \sin \alpha \cos \alpha.$$

Hence, these expressions yield equation

$$mR^2 \ddot{\alpha} - m\omega^2 R^2 \sin \alpha \cos \alpha + mgR \sin \alpha = 0.$$

Since for the static equilibrium position  $\dot{\alpha} = \ddot{\alpha} = 0$ , then, according to the last equation, we have

$$\omega^2 R \sin \alpha \cos \alpha = g \sin \alpha.$$

Hence, the possible equilibrium positions are

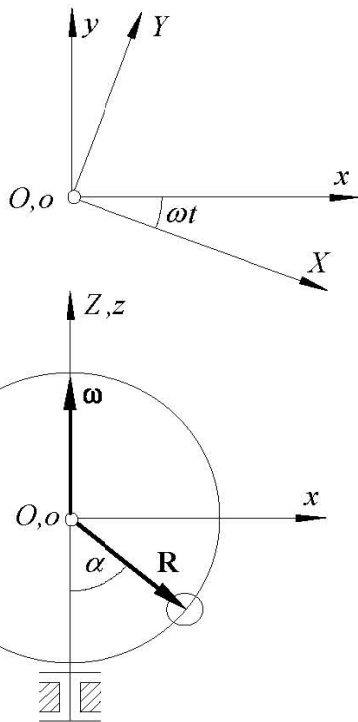


Fig. 98.

of motion

$$\alpha_0 = 0, \alpha_0 = \pi, \alpha_0 = \cos^{-1} \frac{g}{\omega^2 R} = \cos^{-1} \frac{9.81}{10^2 \cdot 0.25} = \pm 66.89^\circ.$$

*Problem 65.* Fig. 99 shows an arm of a robot operating in the horizontal plane. Motion of the arm is controlled by two actuators installed at joints  $O_1$  and  $O_2$ . The actuators produce moments  $M_1$  and  $M_2$ . Derive differential equations of motion of the robot's arm.

Given are:

$I_1$  – moment of inertia of the link 1 about a vertical axis through its centre of gravity  $G_1$ .

$I_2$  – moment of inertia of the link 2 about a vertical axis through its centre of gravity  $G_2$ .

$m_1, m_2$  – masses of the link 1 and 2 respectively.

$a_1, a_2, l_1, l_2$  – dimensions shown in Fig. 99.

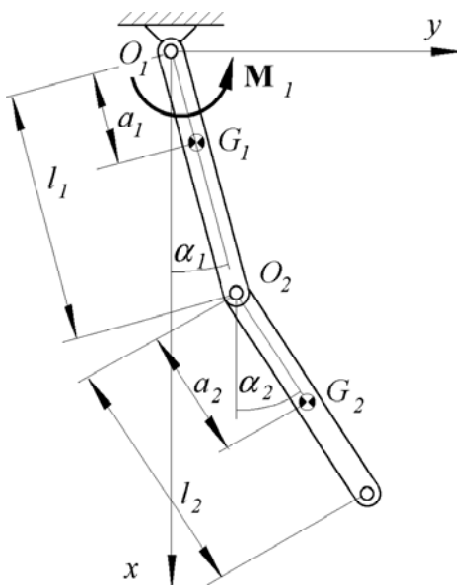


Fig. 99

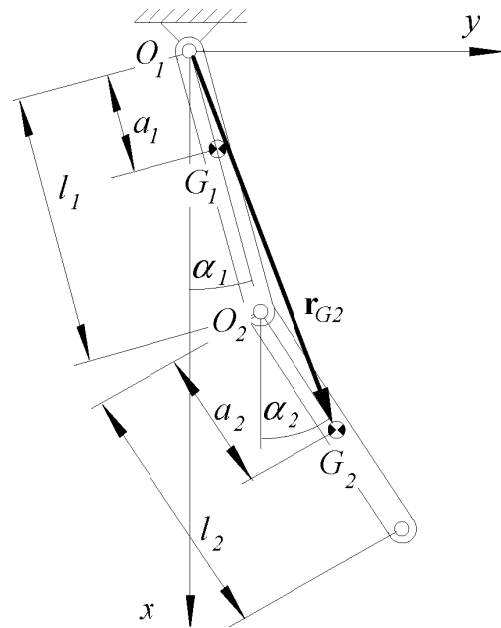


Fig. 100

*Solution.* The system considered has 2 degrees of freedom and the angles  $\alpha_1$  and  $\alpha_2$  may be considered as the generalized coordinates. Hence, Lagrange's equations for this case can be adopted in the following form:

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\alpha}_1} \right) - \left( \frac{\partial T}{\partial \alpha_1} \right) &= Q_1, \\ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\alpha}_2} \right) - \left( \frac{\partial T}{\partial \alpha_2} \right) &= Q_2. \end{aligned}$$

The kinetic energy  $T$  is a sum of kinetic energy of the link 1 and kinetic energy of the link 2.

$$T_1 = \frac{1}{2} (I_1 + m_1 a_1^2) \dot{\alpha}_1^2, \quad T_2 = \frac{1}{2} (m_2 v_{G_2}^2 + I_2 \dot{\alpha}_2^2),$$

where  $v_{G_2}$  is the absolute velocity of the centre of gravity  $G_2$ .

The position vector of centre of gravity  $G_2$  is (Fig. 100)

$$\mathbf{r}_{G_2} = \mathbf{i}(l_1 \cos \alpha_1 + a_2 \cos \alpha_2) + \mathbf{j}(l_1 \sin \alpha_1 + a_2 \sin \alpha_2).$$

Its first derivative yields the velocity of the centre of gravity  $G_2$

$$\mathbf{v}_{G_2} = \mathbf{i}(-l_1 \dot{\alpha}_1 \sin \alpha_1 - a_2 \dot{\alpha}_2 \sin \alpha_2) + \mathbf{j}(l_1 \dot{\alpha}_1 \cos \alpha_1 + a_2 \dot{\alpha}_2 \cos \alpha_2).$$

Squared magnitude of the velocity is



$$v_{G_2}^2 = (-l_1 \dot{\alpha}_1 \sin \alpha_1 - a_2 \dot{\alpha}_2 \sin \alpha_2)^2 + (l_1 \dot{\alpha}_1 \cos \alpha_1 + a_2 \dot{\alpha}_2 \cos \alpha_2)^2$$

$$= l_1^2 \dot{\alpha}_1^2 + a_2^2 \dot{\alpha}_2^2 + 2l_1 a_2 \dot{\alpha}_1 \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2).$$

Upon introducing this expression, one can obtain

$$T_2 = \frac{1}{2} m_2 (l_1^2 \dot{\alpha}_1^2 + a_2^2 \dot{\alpha}_2^2 + 2l_1 a_2 \dot{\alpha}_1 \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2)) + \frac{1}{2} I_2 \dot{\alpha}_2^2.$$

Hence, the total kinetic energy is

$$T = T_1 + T_2 = \frac{1}{2} (I_1 + m_1 a_1^2) \dot{\alpha}_1^2 + \frac{1}{2} m_2 (l_1^2 \dot{\alpha}_1^2 + a_2^2 \dot{\alpha}_2^2 + 2l_1 a_2 \dot{\alpha}_1 \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2)) + \frac{1}{2} I_2 \dot{\alpha}_2^2.$$

Since the robot operates in the horizontal plane, the only non-conservative forces acting on the system are the driving moments.

The virtual work produced by these forces is  $\delta A = (M_1 + M_2) \delta \alpha_1 - M_2 \delta \alpha_2$ .

Therefore, the generalized forces are

$$Q_1 = M_1 + M_2, \quad Q_2 = -M_2.$$

Then

$$\frac{\partial T}{\partial \alpha_1} = -2m_2 l_1 a_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2), \quad \frac{\partial T}{\partial \alpha_2} = 2m_2 l_1 a_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2),$$

$$\frac{\partial T}{\partial \dot{\alpha}_1} = (I_1 + m_1 a_1^2) \dot{\alpha}_1 + m_2 (l_1^2 \dot{\alpha}_1 + l_1 a_2 \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2)), \quad \frac{\partial T}{\partial \dot{\alpha}_2} = m_2 (a_2^2 \dot{\alpha}_2 + l_1 a_2 \dot{\alpha}_1 \cos(\alpha_1 - \alpha_2)) + I_2 \dot{\alpha}_2.$$

Therefore, the final form of equations of motion is

$$[(I_1 + m_1 a_1^2) + m_2 l_1^2] \ddot{\alpha}_1 + 2m_2 l_1 a_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2) = M_1 + M_2,$$

$$(I_2 + m_2 a_2^2) \ddot{\alpha}_2 - 2m_2 l_1 a_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2) = -M_2.$$

*Problem 66.* The angle  $\alpha$  locates the angular position of the stationary slide 4

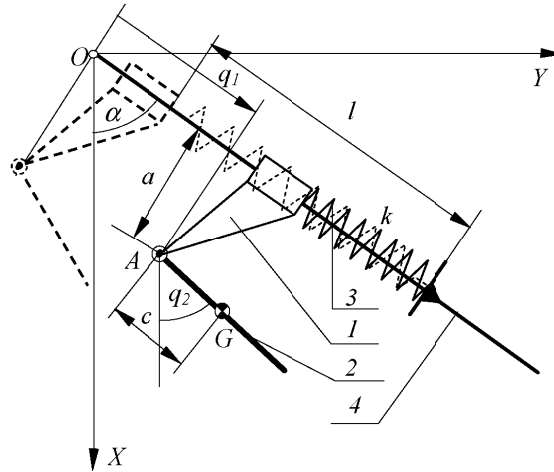


Fig. 101

with respect to the vertical plane  $XY$  of the inertial system of coordinates  $XYZ$  (Fig. 101). The massless link 1 is free to move along this slide and is supported by the spring 3 of stiffness  $k$ . The length of the uncompressed spring is  $l$ . The link 2 is hinged to the link 1 at the point  $A$ . The distance  $c$  locates the position of the centre of gravity  $G$  of the link 2. The link 2 possesses mass  $m$  and its moment of inertia about axis through the centre of gravity  $G$  is  $I$ .

Produce the equations of motion of the system and the expressions for the

generalized coordinates corresponding to the possible equilibrium positions of the system.

*Solution.* This system possesses two degrees of freedom and  $q_1$  and  $q_2$  stand for the generalized independent coordinates. In fact  $q_2$  is an angle of rotation of the link 2 about point  $A$ . Since the link 1 is massless, the total kinetic energy of the system is associated with the link 2 only. The link 2 performs plane motion, hence its kinetic energy is

$$T = \frac{1}{2}mv_G^2 + \frac{1}{2}I\dot{q}_2^2,$$

where  $v_G$  stands for the absolute linear velocity of the centre of gravity  $G$  of the link 2 and  $\dot{q}_2$  is its absolute angular velocity.

The velocity  $v_G$  can be produced by differentiation of the following absolute position vector

$$\mathbf{r}_G = I(q_1 \cos \alpha + a \sin \alpha + c \cos q_2) + J(q_1 \sin \alpha - a \cos \alpha + c \sin q_2).$$

Hence, the wanted velocity is

$$\mathbf{v}_G = \dot{\mathbf{r}}_G = I(\dot{q}_1 \cos \alpha - c \dot{q}_2 \sin q_2) + J(\dot{q}_1 \sin \alpha + c \dot{q}_2 \cos q_2).$$

Then

$$v_G^2 = (\dot{q}_1 \cos \alpha - c \dot{q}_2 \sin q_2)^2 + (\dot{q}_1 \sin \alpha + c \dot{q}_2 \cos q_2)^2.$$

Introduction of these expressions yields the wanted kinetic energy function

$$T = \frac{1}{2}m((\dot{q}_1 \cos \alpha - c \dot{q}_2 \sin q_2)^2 + (\dot{q}_1 \sin \alpha + c \dot{q}_2 \cos q_2)^2) + \frac{1}{2}I\dot{q}_2^2 = \frac{1}{2}m\dot{q}_1^2 + mc\dot{q}_1\dot{q}_2 \sin(\alpha - q_2) + \frac{1}{2}mc^2\dot{q}_2^2 + \frac{1}{2}I\dot{q}_2^2.$$

The elemental work of the force of weight and elastic force of the spring is

$$\delta A = mg \cos \alpha \delta q_1 - kq_1 \delta q_1 - mgc \sin q_2 \delta q_2.$$

Therefore, the generalized forces are

$$Q_1 = mg \cos \alpha - kq_1, Q_2 = -mgc \sin q_2.$$

Since the system is of two degrees of freedom and the generalized coordinates are  $q_1$  and  $q_2$ , one can obtain the following Lagrange's equations:

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_1} \right) - \frac{\partial T}{\partial q_1} = Q_1, \quad \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_2} \right) - \frac{\partial T}{\partial q_2} = Q_2.$$

Taking into account the formulas obtained above, we have the equations of motion

$$m\ddot{q}_1 + mc \sin(\alpha - q_2) \ddot{q}_2 - mc \cos(\alpha - q_2) \dot{q}_2^2 - mg \cos \alpha + kq_1 = 0, \\ (I + mc^2)\ddot{q}_2 + mc \sin(\alpha - q_2) \dot{q}_1 + mgc \sin q_2 = 0.$$

The above set of equations allows the equilibrium position of the system to be determined. If  $q_1$  and  $q_2$  are constant,

$$mg \cos \alpha - kq_1 = 0, \\ mgc \sin q_2 = 0.$$

$$\text{Hence, } q_1 = \frac{mg \cos \alpha}{k}, \quad q_2 = 0.$$

*Problem 67.* The circular slide of radius  $R$  is free to rotate about the horizontal axis  $Y$  of the inertial system of coordinates  $XYZ$  (Fig. 102). Its moment of inertia about that axis is  $I$ . The body 2, which can be considered as a particle of mass  $m$ , can move along the slide without friction. System of coordinates  $xyz$ , shown in Fig. 102, is rigidly attached to the slide  $l$ .

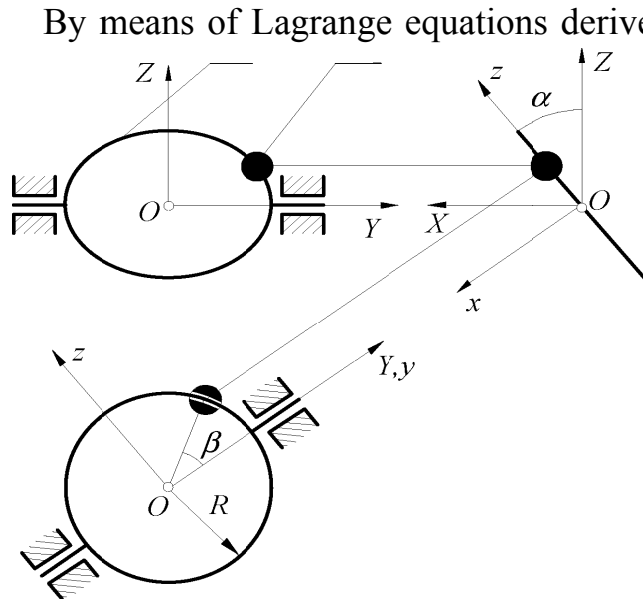


Fig. 102

and kinetic energy of the slide  $T = T_1 + T_2$ , where  $T_1$  is a kinetic energy of the particle and  $T_2$  is a kinetic energy of the slide.

$T_1 = \frac{mv^2}{2}$ , where  $v$  is an absolute velocity of a particle. Then  $\mathbf{v}_a = \mathbf{v}_r + \mathbf{v}_{tr}$ , where  $\mathbf{v}_r$  is a relative velocity and  $\mathbf{v}_{tr}$  is a transport velocity of the particle.

Relative motion is a motion of the particle along the slide. Hence,  $v_r = R\dot{\beta}$ . Transport motion is a motion of the slide. Therefore,  $v_{tr} = R\dot{\alpha} \sin \beta$ . But  $\mathbf{v}_r \perp \mathbf{v}_{tr}$ , so  $v_a^2 = R^2(\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2)$  and  $T_1 = \frac{mR^2(\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2)}{2}$ .

Slide is in rotational motion, then  $T_2 = \frac{I\dot{\alpha}^2}{2}$ .

The total kinetic energy is

$$T = \frac{mR^2(\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2)}{2} + \frac{I\dot{\alpha}^2}{2}.$$

By derivation one can obtain

$$\frac{\partial T}{\partial \dot{\alpha}} = I\dot{\alpha} + mR^2\dot{\alpha} \sin^2 \beta, \quad \frac{\partial T}{\partial \dot{\beta}} = mR^2\dot{\beta}, \quad \frac{\partial T}{\partial \alpha} = 0, \quad \frac{\partial T}{\partial \beta} = mR^2\dot{\alpha}^2 \sin \beta \cos \beta.$$

Now determine the generalized forces. Considering elemental work done by a gravity force in virtual displacement  $\delta\alpha$ , we have

$$\delta A_\alpha = mgR \sin \alpha \sin \beta \delta\alpha.$$

The elemental work done by a gravity force in virtual displacement  $\delta\beta$  is

$$\delta A_\beta = -mgR \cos \alpha \cos \beta \delta\beta.$$

Thus,  $Q_\alpha = mgR \sin \alpha \sin \beta$  and  $Q_\beta = -mgR \cos \alpha \cos \beta$ .

Finally we have the differential equations of motion of the system

$$(I + mR^2 \sin^2 \beta)\ddot{\alpha} + 2mR^2\dot{\alpha}\dot{\beta} \sin \beta \cos \beta - mgR \sin \alpha \sin \beta = 0, \\ mR^2\ddot{\beta} - mR^2\dot{\alpha}^2 \sin \beta \cos \beta + mgR \cos \alpha \cos \beta = 0.$$

By means of Lagrange equations derive the differential equations of motion of the system along the generalized coordinates  $\alpha$  and  $\beta$ .

*Solution.* The system considered has 2 degrees of freedom and the angles  $\alpha$  and  $\beta$  may be considered as the generalized coordinates. Hence, Lagrange's equations for this case can be adopted in the following form:

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\alpha}} \right) - \left( \frac{\partial T}{\partial \alpha} \right) = Q_\alpha, \\ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\beta}} \right) - \left( \frac{\partial T}{\partial \beta} \right) = Q_\beta.$$

The kinetic energy  $T$  is a sum of kinetic energy of the body (particle)

## 8. PROBLEMS FOR SELF-STUDY TRAINING

### 8.1. Integration of Differential Equations of the Particle Motion under the Action of Constant Forces

A body moves from the point  $A$  along a plane  $AB$  of length  $l$  during  $\tau$  s. (Fig. 103). Its initial velocity is  $v_A$ . Coefficient of sliding friction is  $f$ . At the point  $B$  the body leaves a plane with a velocity  $v_B$  and then falls with a velocity  $v_C$  at a point  $C$  moving in the air  $T$  s.

Determine specified quantities. Consider the body as a material particle neglecting the resistance of the air.

#### Variants 1—5 (Fig. 103, scheme 1).

Variant 1. Given are:  $\alpha = 30^\circ$ ;  $v_A = 0$ ;  $f = 0,2$ ;  $l = 10$  m;  $\beta = 60^\circ$ . Determine  $\tau$  and  $h$ .

Variant 2. Given are:  $\alpha = 15^\circ$ ;  $v_A = 2$  m/s;  $f = 0,2$ ;  $h = 4$  m;  $\beta = 45^\circ$ . Determine  $l$  and equation of the path along  $BC$ .

Variant 3. Given are:  $\alpha = 30^\circ$ ;  $v_A = 2,5$  m/s;  $f \neq 0$ ;  $l = 8$  m;  $d = 10$  m;  $\beta = 60^\circ$ . Determine  $v_B$  and  $\tau$ .

Variant 4. Given are:  $v_A = 0$ ;  $\tau = 2$  s;  $l = 9,8$  m;  $\beta = 60^\circ$ ;  $f = 0$ . Determine  $\alpha$  and  $T$ .

Variant 5. Given are:  $\alpha = 30^\circ$ ;  $v_A = 0$ ;  $l = 9,8$  m;  $\tau = 3$  s;  $\beta = 45^\circ$ . Determine  $f$  and  $v_C$ .

#### Variants 6—10 (Fig. 103, scheme 2).

Variant 6. Given are:  $\alpha = 20^\circ$ ;  $f = 0,1$ ;  $\tau = 0,2$  s;  $h = 40$  m;  $\beta = 30^\circ$ . Determine  $l$  and  $v_C$ .

Variant 7. Given are:  $\alpha = 15^\circ$ ;  $f = 0,1$ ;  $v_A = 16$  m/s;  $l = 5$  m;  $\beta = 45^\circ$ . Determine  $v_B$  and  $T$ .

Variant 8. Given are:  $v_A = 21$  m/s;  $f = 0$ ;  $\tau = 0,3$  s;  $v_B = 20$  m/c;  $\beta = 60^\circ$ . Determine  $\alpha$  and  $d$ .

Variant 9. Given are:  $\alpha = 15^\circ$ ;  $\tau = 0,3$  s;  $f = 0,1$ ;  $h = 30\sqrt{2}$  m;  $\beta = 45^\circ$ . Determine  $v_B$  and  $v_A$ .

Variant 10. Given are:  $\alpha = 15^\circ$ ;  $f = 0$ ;  $v_A = 12$  m/s;  $d = 50$  m;  $\beta = 60^\circ$ . Determine  $\tau$  and equation of the path along  $BC$ .

#### Variants 11—15 (Fig. 103, scheme 3, $f = 0$ , $M$ is a mass of a body).

Variant 11. Given are:  $\alpha = 30^\circ$ ;  $P \neq 0$ ;  $l = 40$  m;  $v_A = 0$ ;  $v_B = 4,5$  m/s;  $d = 3$  m. Determine  $\tau$  and  $h$ .

Variant 12. Given are:  $\alpha = 30^\circ$ ;  $P = 0$ ;  $l = 40$  m;  $v_B = 4,5$  m/s;  $h = 1,5$  m. Determine  $v_A$  and  $d$ .

Variant 13. Given are:  $\alpha = 30^\circ$ ;  $M = 400$  kg;  $v_A = 0$ ;  $\tau = 20$  s;  $d =$

3 m;  $h = 1,5$  m. Determine  $P$  and  $l$ .

Variant 14. Given are:  $\alpha = 30^\circ$ ;  $M = 400$  kg;  $P = 2,2$  kN;  $v_A = 0$ ;  $l = 40$  m;  $d = 5$  m. Determine  $v_B$  and  $v_C$ .

Variant 15. Given are:  $\alpha = 30^\circ$ ;  $v_A = 0$ ;  $P = 2$  kN;  $l = 50$  m;  $h = 2$  m;  $d = 4$  m. Determine  $T$  and  $M$ .

**Variants 16—20 (Fig. 103, scheme 4).**

Variant 16. Given are:  $\alpha = 30^\circ$ ;  $v_A = 1$  m/s;  $l = 3$  m;  $f = 0,2$ ;  $d = 2,5$  m. Determine  $h$  and  $T$ .

Variant 17. Given are:  $\alpha = 45^\circ$ ;  $l = 6$  m;  $v_B = 2v_A$ ;  $\tau = 1$  s;  $h = 6$  m. Determine  $d$  and  $f$ .

Variant 18. Given are:  $\alpha = 30^\circ$ ;  $l = 2$  m;  $v_A = 0$ ;  $f = 0,1$ ;  $d = 3$  m. Determine  $h$  and  $\tau$ .

Variant 19. Given are:  $\alpha = 15^\circ$ ;  $l = 3$  m;  $v_B = 3$  m/s;  $f \neq 0$ ;  $\tau = 1,5$  s;  $d = 2$  m. Determine  $v_A$  and  $h$ .

Variant 20. Given are:  $\alpha = 45^\circ$ ;  $v_A = 0$ ;  $f = 0,3$ ;  $d = 2$  m;  $h = 4$  m. Determine  $l$  and  $\tau$ .

**Variants 21—25 (Fig. 103, scheme 5).**

Variant 21. Given are:  $\alpha = 30^\circ$ ;  $f = 0,1$ ;  $v_A = 1$  m/s;  $\tau = 1,5$  s;  $h = 10$  m. Determine  $v_B$  and  $d$ .

Variant 22. Given are:  $v_A = 0$ ;  $\alpha = 45^\circ$ ;  $l = 10$  m;  $\tau = 2$  s. Determine  $f$  and equation of the path along  $BC$ .

Variant 23. Given are:  $f = 0$ ;  $v_A = 0$ ;  $l = 9,81$  m;  $\tau = 2$  s;  $h = 20$  m. Determine  $\alpha$  and  $T$ .

Variant 24. Given are:  $v_A = 0$ ;  $\alpha = 30^\circ$ ;  $f = 0,2$ ;  $l = 10$  m;  $d = 12$  m. Determine  $\tau$  and  $h$ .

Variant 25. Given are:  $v_A = 0$ ;  $\alpha = 30^\circ$ ;  $f = 0,2$ ;  $l = 6$  m;  $h = 4,5$  m. Determine  $\tau$  and  $v_C$ .

**Variants 26—30 (Fig. 103, scheme 6).**

Variant 26. Given are:  $v_A = 7$  m/s;  $f = 0,2$ ;  $l = 8$  m;  $h = 20$  m. Determine  $d$  and  $v_C$ .

Variant 27. Given are:  $v_A = 4$  m/s;  $f = 0,1$ ;  $\tau = 2$  s;  $d = 2$  m. Determine  $v_B$  and  $h$ .

Variant 28. Given are:  $v_B = 3$  m/s;  $f = 0,3$ ;  $l = 3$  m;  $h = 5$  m. Determine  $v_A$  and  $T$ .

Variant 29. Given are:  $v_A = 3$  m/s;  $v_B = 1$  m/s;  $l = 2,5$  m;  $h = 20$  m. Determine  $f$  and  $d$ .

Variant 30. Given are:  $f = 0,25$ ;  $l = 4$  m;  $d = 3$  m;  $h = 5$  m. Determine  $v_A$  and  $\tau$ .

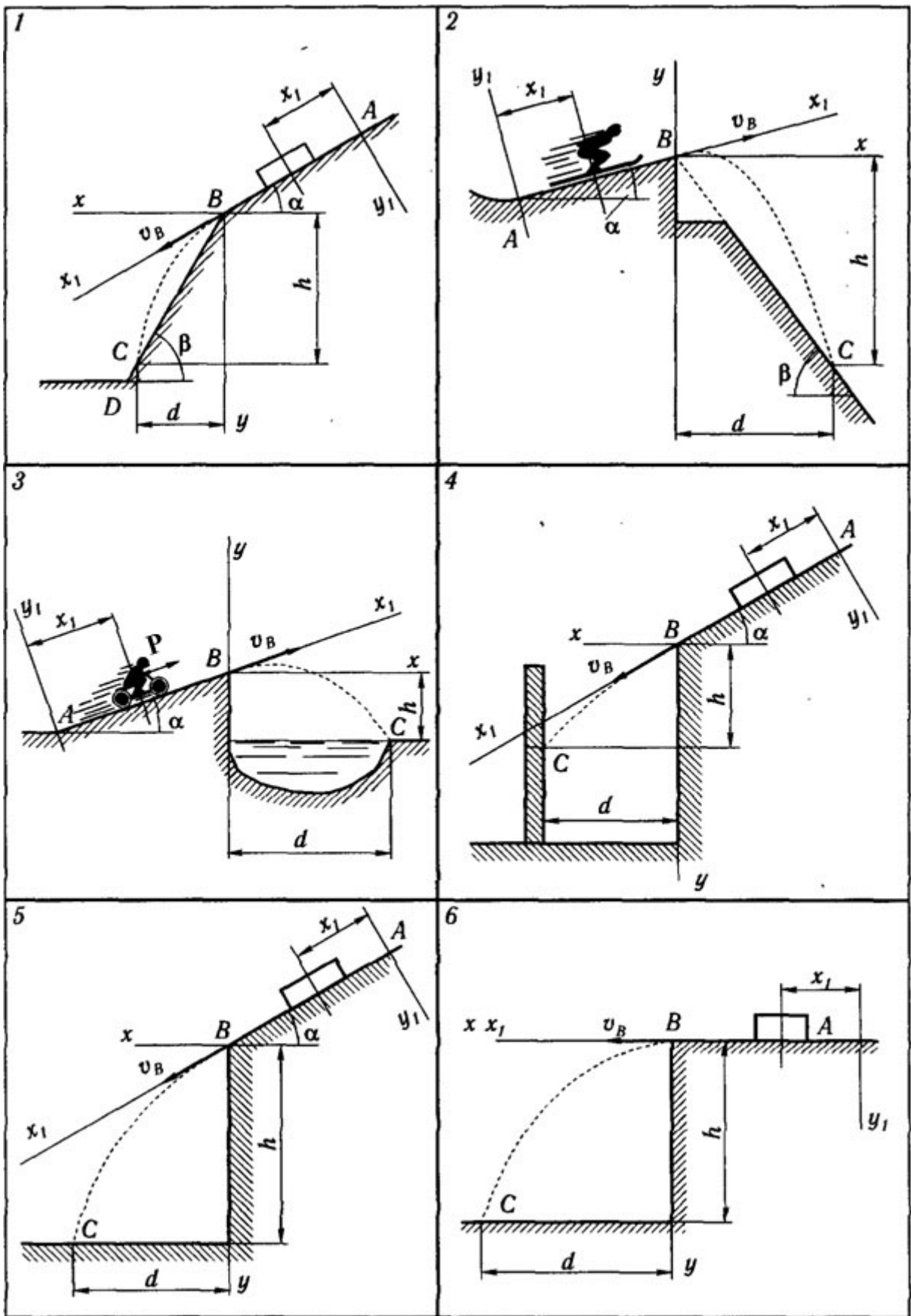


Fig. 103

*Example*

In order to protect ditches from the slide-rocks there is a ledge  $DC$  in the railway rocky cuttings. Taking into account the possibility of the motion of a stone from the highest point  $A$  and assuming its initial velocity to be  $v_0 = 0$ , determine the minimal width of the ledge  $b$  and falling velocity  $v_C$ . The stone moves along a slope  $AB$  of length  $l$  during  $\tau$  s. The angle  $\alpha$  is given. Coefficient of sliding friction  $f$  is constant. Neglect the resistance of the air.

Given are:  $v_A = 0$ ;  $\alpha = 60^\circ$ ;  $l = 4$  m;  $\tau = 1$  s;  $f \neq 0$ ;  $h = 5$  m;  $\beta = 75^\circ$ .  
Determine  $b$  and  $v_C = 0$ .

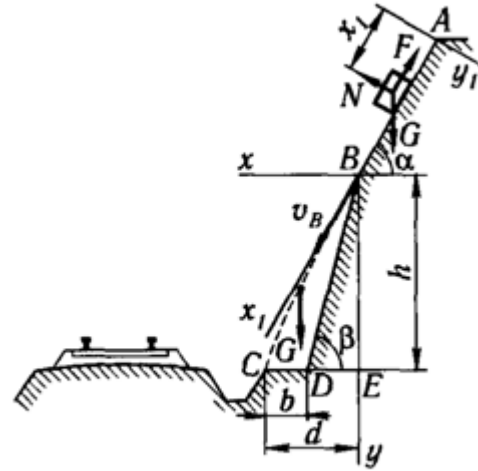


Fig. 104

*Solution.* Consider the motion of a stone along  $AB$ . Assuming the stone as a material particle, show all acting forces: weight  $G$ , normal reaction  $N$  and a force of sliding friction  $F$ . Work out differential equation of motion of the stone on section  $AB$  (Fig. 104):

$$m\ddot{x}_1 = \sum X_{i1}; \quad m\ddot{x}_1 = G \sin \alpha - F.$$

The force of friction is

$$F = fN,$$

where  $N = G \cos \alpha$ .

Therefore,

$$m\ddot{x}_1 = G \sin \alpha - fG \cos \alpha \text{ or } \ddot{x}_1 = g \sin \alpha - fg \cos \alpha.$$

Integrating this differential equation twice, we obtain

$$\begin{aligned} \dot{x}_1 &= g(\sin \alpha - f \cos \alpha)t + C_1; \\ x_1 &= \left[ \frac{g(\sin \alpha - f \cos \alpha)}{2} \right] t^2 + C_1 t + C_2. \end{aligned}$$

In order to determine constants of integration, make use of initial conditions: at  $t = 0$ ,  $x_{10} = 0$  and  $\dot{x}_{10} = 0^*$ .

Compose equations by integrating, for  $t = 0$ :

$$\dot{x}_{10} = C_1; \quad x_{10} = C_2.$$

Determine constants:

$$C_1 = 0, \quad C_2 = 0.$$

Then

$$\dot{x}_1 = g(\sin \alpha - f \cos \alpha)t; \quad x_1 = \left[ \frac{g(\sin \alpha - f \cos \alpha)}{2} \right] t^2.$$

For instant  $\tau$ , when the stone leaves rectilinear section,

$$\dot{x}_1 = v_B; \quad x_1 = l,$$

i.e.,

$$v_B = g(\sin \alpha - f \cos \alpha)\tau;$$

$$l = \left[ \frac{g(\sin \alpha - f \cos \alpha)}{2} \right] \tau^2,$$

whence

$$v_B = \frac{2l}{\tau},$$

i.e.,

$$v_B = \frac{2 \cdot 4}{1} = 8 \text{ m/s}.$$

Consider the motion of a stone on curvilinear section  $BC$ . There is only a force of weight  $\mathbf{G}$  acting on a stone here. Derive differential equations of its motion:

$$m\ddot{x} = 0; \quad m\ddot{y} = G.$$

Initial conditions of the problem: at  $t = 0$ ,

$$\begin{aligned} x_0 &= 0; & y_0 &= 0; \\ \dot{x}_0 &= v_B \cos \alpha; & \dot{y}_0 &= v_B \sin \alpha. \end{aligned}$$

Integrating these differential equations twice, we obtain

$$\begin{aligned} \dot{x} &= C_3; & \dot{y} &= gt + C_4; \\ x &= C_3t + C_5; & y &= gt^2/2 + C_4t + C_6. \end{aligned}$$

Write these equations for  $t = 0$ :

$$\begin{aligned} \dot{x}_0 &= C_3; & \dot{y}_0 &= C_4; \\ x_0 &= C_5; & y_0 &= C_6. \end{aligned}$$

Whence,

$$\begin{aligned} C_3 &= v_B \cos \alpha; & C_4 &= v_B \sin \alpha; \\ C_5 &= 0; & C_6 &= 0. \end{aligned}$$

Equations for the projections of velocity of a stone are

$$\dot{x} = v_B \cos \alpha; \quad \dot{y} = gt + v_B \sin \alpha,$$

and equations of its motion have the following form:

$$x = v_B t \cos \alpha; \quad y = gt^2/2 + v_B t \sin \alpha.$$

By excluding parameter  $t$  from equations of motion one can derive equation of the path of the stone. Determine  $t$  from the first equation and then substitute its value into the second one. We have

$$y = gx^2/(2v_B^2 \cos^2 \alpha) + xt g \alpha.$$

At point  $C$ ,  $y = h = 5 \text{ m}$ ,  $x = d$ .

Determining  $d$  from equation of the path we have

$$d_1 = 2,11 \text{ m}, \quad d_2 = -7,75 \text{ m}.$$

Since equation of the path is a branch of parabola with positive abscissas of its points, then  $d = 2,11 \text{ m}$ . The minimal width  $CD$  is

$$b = d - ED = d - h/\text{tg}75^\circ, \quad \text{or } b = 0,77 \text{ m}.$$

By making use of equation of motion  $x = v_B t \cos \alpha$ , we determine the time  $T$  of motion of the stone from the point  $B$  to the point  $C$ :



$$T = 0,53 \text{ s.}$$

Velocity of the stone at the point  $C$  may be determined by calculating their projections on coordinate axes

$$\dot{x} = v_B \cos \alpha; \quad \dot{y} = gt + v_B \sin \alpha.$$

Finally we have

$$v = \sqrt{\dot{x}^2 + \dot{y}^2}.$$

For instant, when the stone is at the point  $C$ , we have  $t = T = 0,53 \text{ s}$  and

$$v_C = \sqrt{(v_B \cos \alpha)^2 + (gT + v_B \sin \alpha)^2}, \text{ or } v_C = 12,8 \text{ m/s.}$$

\* The constants of integration  $C_1 - C_6$  can be determined by considering initial conditions for the first and the second sections of the motion. Nevertheless, sometimes it is easier to use boundary conditions for different instants.

## 8.2. Application of the Theorem of the Change in Kinetic Energy to Study of the Motion of a System

Mechanical system starts moving from the state of rest under the action of the forces of weight; initial state of the system is shown in Figs. 106-108. Determine acceleration of a body  $1$  and its velocity when it has travelled a distance  $s$ . Take into account a force of sliding friction (variants 1-3, 5, 6, 8-12, 17—23, 28-30) and the rolling friction of a body 3 (variants 2, 4, 6-9, 11, 13-15, 20, 21, 24, 27, 29). Neglect other forces of resistance and masses of the strings. Assume the strings as inextensible. Make use of the following designations:  $m_1, m_2, m_3, m_4$  are the masses of the bodies  $1, 2, 3, 4$ ;  $R_2, r_2, R_3, r_3$ — radii of respective circumferences;  $i_2, i_3$ — radii of gyration of the bodies 2 and 3 with respect to axes perpendicular to the plane of figure trough their centers of gravity;  $\alpha, \beta$ — angles of inclination of planes to horizon;  $f$ —coefficient of sliding friction;  $\delta$ — coefficient of rolling friction.

The necessary data are represented in tab.1. Assume pulleys and blocks for which the radii of gyration are not given as homogeneous disks. Inclined strings are parallel to respective inclined planes.

*Example.* Given are:  $m_1$  is a mass of the weight  $1$ ,  $m_2 = 2m_1$ ,  $m_3 = m_1$ ,  $m_4 = 0,5m_1$ ,  $m_5 = 20m_1$ ,  $R_2 = R_3 = 12 \text{ cm}$ ,  $r_2 = 0,5R_2$ ,  $r_3 = 0,75R_3$ ,  $R_5 = 20 \text{ cm}$ ,  $AB = l = 4R_3$ ,  $i_2 = 8 \text{ cm}$ ,  $i_3 = 10 \text{ cm}$ ,  $\alpha = 30^\circ$ ,  $f = 0,1$ ,  $\delta = 0,2 \text{ cm}$ ,  $s = 0,06\pi \text{ m}$ .

Neglect rolling friction of a body 2, masses of the link  $BC_5$  and the slide  $B$ .

Connecting rod 4 is a thin homogeneous rod. Roller 5 is a homogeneous solid cylinder. Initial state of a system is shown in Fig. 105a.

Determine  $v_1$ —velocity of the weight in a final position and its acceleration.

*Solution.* Apply the theorem of the change in kinetic energy of a system:

$$T - T_0 = \sum A_i^E + \sum A_i^J,$$

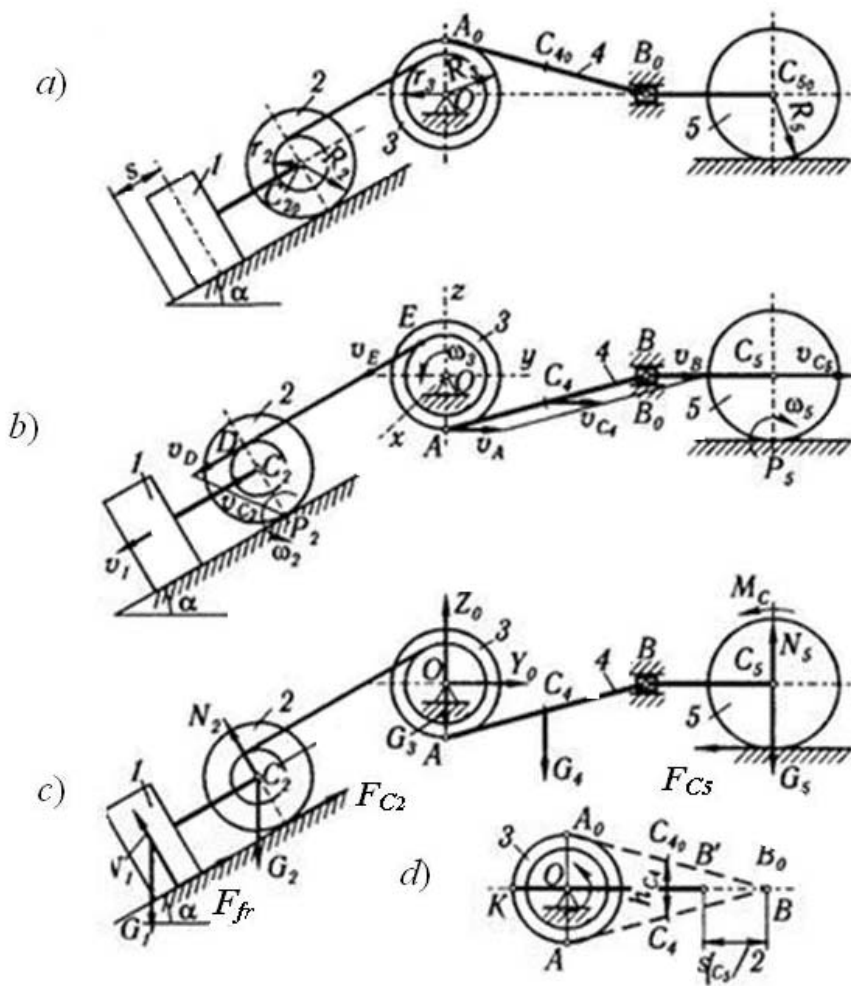


Fig. 105

In order to define kinetic energy  $T$  and the sum of the work done by all the external forces, depict the system in a final position (Fig. 105b,c).

Write down relations between velocities and displacements of the points of a system. Express all velocities and displacements through velocity and displacement of the weight  $l$ .

Velocity of the center of mass of the roller 2 equals velocity of the weight  $l$ :  $v_{C_2} = v_1$ .

The instantaneous center of zero velocity of the roller 2 is located at point  $P_2$ . Its angular velocity is

$$\omega_2 = \frac{v_{C_2}}{C_2P_2} \text{ or } \omega_2 = \frac{v_1}{R_2}.$$

Velocity of the point  $D$  is

$$v_D = \omega_2 DP_2, \text{ i.e.,} \\ v_D = \frac{v_1(R_2 + r_2)}{R_2}.$$

It is obvious that  $v_E = v_D$ . But  $v_E = \omega_3 r_3$ , hence,

$$\omega_3 r_3 = \frac{v_1}{R_2} (R_2 + r_2).$$

Since  $R_2 = 2r_2$ , then

where  $T$  and  $T_0$  are kinetic energy of a system in a final and initial position;  $\sum A_i^E$  is a sum of the work done by all the external forces acting on the system during its displacement from initial to final position;  $\sum A_i^J$  is a sum of the work done by all the internal forces in that displacement.

For systems in question which consist of solids connected by inextensible thread and rods

$$\sum A_i^J = 0.$$

Since the system in initial position is in a state of rest,  $T_0 = 0$ .

Hence, we have  $T = \sum A_i^E$ .

$$\omega_3 r_3 = \frac{3}{2} v_1,$$

whence  $\omega_3 = \frac{3}{2} \frac{v_1}{r_3}$ .

Taking into account that

$$\omega_3 = \frac{d\varphi_3}{dt}, v_1 = \frac{ds}{dt},$$

we have

$$\frac{d\varphi_3}{dt} = \frac{3}{2r_3} \frac{ds}{dt}, \text{ or } d\varphi_3 = \frac{3}{2r_3} ds.$$

By integrating one can obtain

$$\varphi_3 = \frac{3}{2} \frac{s}{r_3}.$$

When the weight  $l$  travels a distance  $s = 0,06\pi \text{ m}$ , the pulley 3 turns through the angle  $\varphi_3$ :

$$\varphi_3 = \frac{3}{2} \frac{s}{r_3} = \frac{3}{2} \frac{0,06\pi}{0,09} = \pi.$$

At this angle of rotation of the pulley 3 on  $180^\circ$  its point  $A_0$  moves to the final location  $A$ , and connecting rod 4 moves from initial location  $A_0B_0$  to the final location  $AB$ .

The roller 5 moves to the left at the angle of rotation of the pulley 3 equal to  $\frac{\pi}{2}$ , and it moves to the right at the angle of rotation equal to  $\pi$ . Hence, the final location of the roller 5 coincides with its initial location.

Thus, the final location of all the parts of a system is defined completely (Fig. 105b).

Determine kinetic energy of a system in the final position as a sum of kinetic energy of the bodies 1, 2, 3, 4, 5:

$$T = T_1 + T_2 + T_3 + T_4 + T_5.$$

Kinetic energy of the load 1, which is in translational motion, is

$$T_1 = \frac{m_1 v_1^2}{2}.$$

Kinetic energy of the roller 2, which is in a plane motion, is

$$T_2 = \frac{m_2 v_{C2}^2}{2} + \frac{J_{2\xi} \omega_2^2}{2},$$

where  $J_{2\xi}$  is a moment of inertia of the roller 2 with respect to its longitudinal central axis  $C_{2\xi}$ :

$$J_{2\xi} = m_2 i_2^2.$$

Then we find

$$T_2 = \frac{m_2 v_1^2}{2} + \frac{m_2 i_2^2}{2R_2^2} v_1^2 = \frac{1}{2} m_2 \left( 1 + \frac{i_2^2}{R_2^2} \right) v_1^2.$$

Kinetic energy of the body 3, which rotates around the axis  $Ox$ , is

$$T_3 = \frac{1}{2} J_{3x} \omega_3^2,$$

where  $J_{3x}$  is a moment of inertia of the block 3 with respect to the axis  $Ox$ :

$$J_{3x} = m_3 i_3^2.$$

Then for the body 3 we obtain

$$T_3 = \frac{m_3 i_3^2}{2} \left( \frac{3}{2} \frac{v_1}{r_3} \right)^2 = \frac{9}{8} m_3 \frac{i_3^2}{r_3^2} v_1^2.$$

Kinetic energy of the connecting rod 4, which is in a plane motion, is

$$T_4 = \frac{m_4 v_{C4}^2}{2} + \frac{J_{4\xi} \omega_4^2}{2},$$

where  $v_{C4}$  is a velocity of the center of mass of the connecting rod 4;  $J_{4\xi}$  is its moment of inertia with respect to the central axis  $C_{4\xi}$ .

In order to determine  $v_{C4}$  and  $\omega_4$ , find location of the instantaneous center of zero velocity of the connecting rod 4. Since points  $A$  and  $B$  at this instant are parallel, the instantaneous center of zero velocity of the connecting rod 4 lies in infinity. Hence, its angular velocity at the given instant is  $\omega_4 = 0$ , and velocities of all the points are parallel and equal. Thus, kinetic energy of the connecting rod 4 is

$$T_4 = \frac{m_4 v_{C4}^2}{2},$$

where  $v_{C4} = v_A$ .

Linear velocity of the point  $A$  of the body 3 is

$$v_A = \omega_3 R_3, \text{ or } v_A = \frac{3}{2} R_3 v_1 / r_3.$$

Since  $r_3 = 3/4 R_3$ , we have  $v_A = 2v_1$ .

But  $v_{C4} = v_A$ ,  $v_{C4} = 2v_1$ .

So, the expression for kinetic energy of the connecting rod 4 has the following form:

$$T_4 = \frac{1}{2} m_4 (2v_1)^2 = 2m_4 v_1^2.$$

Kinetic energy of the roller 5, which is in a plane motion, is

$$T_5 = \frac{m_5 v_{C5}^2}{2} + \frac{J_{5\xi} \omega_5^2}{2},$$

where  $v_{C5}$  is a velocity of the center of mass  $C_5$  of the roller 5;  $J_{5\xi}$  — its moment of inertia (as a homogeneous solid cylinder) with respect to its longitudinal central axis  $C_{5\xi}$ ;  $J_{5\xi} = \frac{m_5 R_5^2}{2}$ ;  $\omega_5$  — its angular velocity.

Since the roller moves without slipping, its instantaneous center of zero velocity is at point  $P_5$ . Then

$$\omega_5 = \frac{v_{C5}}{R_5}.$$

Hence,

$$T_5 = \frac{m_5 v_{C5}^2}{2} + \frac{m_5 R_5^2 v_{C5}^2}{2 \cdot 2R_5^2} = \frac{3}{4} m_5 v_{C5}^2.$$

As far as the link  $BC_5$  is in a plane motion,  $v_{C5} = v_B$ . But  $v_B = v_{C4} = 2v_1$ , then  $v_{C5} = 2v_1$ .

Therefore, kinetic energy of the roller 5 is

$$T_5 = \frac{3}{4} m_5 (2v_1)^2 = 3m_5 v_1^2.$$

The total kinetic energy of the system will be

$$T = \frac{m_1 v_1^2}{2} + m_2 (1 + i_2^2/R_2^2) v_1^2/2 + \frac{9}{8} m_3 v_1^2 i_3^2/r_3^2 + 2m_4 v_1^2 + 3m_5 v_1^2.$$

Substituting values of the given masses, we obtain

$$T = m_1 v_1^2 [1 + 2(1 + i_2^2/R_2^2) + \frac{9}{4} i_3^2/r_3^2 + 2 + 120]/2, \text{ or}$$

$$T = 129m_1 v_1^2/2.$$

Define the sum of the work done by all the external forces acting on a system in its specified displacement. Depict all the external forces (Fig. 105c).

Work done by the weight  $G_1$  is

$$A_{G_1} = G_1 h_1 = m_1 g s \sin \alpha.$$

Work done by frictional force  $F_{fr}$  is

$$A_{F_{fr}} = -F_{fr} s.$$

As far as  $F_{fr} = fN_1 = fG_1 \cos \alpha$ , then

$$A_{F_{fr}} = -f m_1 g s \cos \alpha.$$

Work done by the weight  $G_2$  is

$$A_{G_2} = G_2 h_{C_2} = m_2 g s \sin \alpha.$$

Work done by forces of traction  $F_{C_2}, F_{C_5}$  of the rollers 2 and 5 is zero since these forces are applied at their instantaneous centers of zero velocity.

Work done by the weight  $G_4$  is

$$A_{G_4} = G_4 h_{C_4},$$

where  $h_{C_4}$  is a vertical displacement of the center of gravity  $C_4$  of the connecting rod 4 from initial location to its final position (Fig. 105d):

$$h_{C_4} = R_3, A_{G_4} = m_4 g R_3.$$

Work done by the rolling friction of the roller 5 is

$$A_{M_C} = -M_C \varphi_5,$$

where  $M_C = \delta N_5 = \delta G_5$  is a moment of a couple of the resisting forces to rolling of the roller 5;  $\varphi_5$  is an angle of its rotation.

Since roller 5 moves without slipping, angle of its rotation is

$$\varphi_5 = s_{C_5}/R_5,$$

where  $s_{C_5}$  is a displacement of the center of gravity  $C_5$  of the roller 5.

In this example the work of the mentioned couple is calculated as a sum of the work done by this couple at the rotation of the body 3 on the angle  $\pi/2$  to the left and when the body 3 turns on angle  $\pi/2$  to the right once more.

The displacement of the center of gravity  $C_5$  of the roller 5 equals displacement of the slider  $B$  to the left and to the right:

$$s_{C5} = 2(B_0B').$$

Determine the displacement  $B_0B'$  at the rotation of body 3 on angle  $\pi/2$ . Choose fixed point  $K$  of the plane as a reference (Fig. 105d). At this rotation of the body 3 the connecting rod will move from position  $A_0B_0$  to position  $KB'$ . Then

$$B_0B' = KB_0 - KB',$$

$$\text{where } KB_0 = KO + OB_0 = R_3 + \sqrt{(A_0B_0)^2 - (A_0O)^2} = R_3 + \sqrt{l^2 - R_3^2},$$

$$KB' = l = 4R_3.$$

Hence,

$$B_0B' = R_3 + \sqrt{l^2 - R_3^2} - l = R_3 + \sqrt{(4R_3)^2 - R_3^2} - 4R_3 = 0,88R_3.$$

The total angle of rotation of the roller 5 is

$$\varphi_5 = 1,76R_3/R_5.$$

$$\text{Then } A_{M_C} = -\delta m_5 g \cdot 1,76R_3/R_5.$$

The total sum of the work done by all the external forces is

$$\sum A_i^E = m_1 g s \sin \alpha - f m_1 g s \cos \alpha + m_2 g s \sin \alpha + m_4 g R_3 - \delta m_5 g \cdot 1,76R_3/R_5.$$

By substituting the given values of masses we obtain

$$\sum A_i^E = m_1 g s (\sin \alpha - f \cos \alpha + 2 \sin \alpha + \frac{R_3}{2s} - \frac{\delta \cdot 20 \cdot 1,76R_3}{R_5 s})$$

$$\text{or } \sum A_i^E = 1,51 m_1 g s.$$

According to the theorem of the change in kinetic energy of a system equate the values  $T$  and  $\sum A_i^E$ :

$$129 \cdot \frac{m_1 v_1^2}{2} = 1,51 m_1 g s,$$

whence

$$v_1 = 0,21 \text{ m/s.}$$

In order to define acceleration of the weight, make use of the theorem in differential form:

$$dT = \sum d A_i^E.$$

Therefore,

$$129 \cdot \frac{m_1}{2} 2v_1 dv_1 = 1,51 m_1 g ds,$$

$$\text{whence } \frac{v_1 dv_1}{ds} = \frac{1,51}{129} g \text{ or } a_1 = 0,115 \text{ m/s}^2.$$

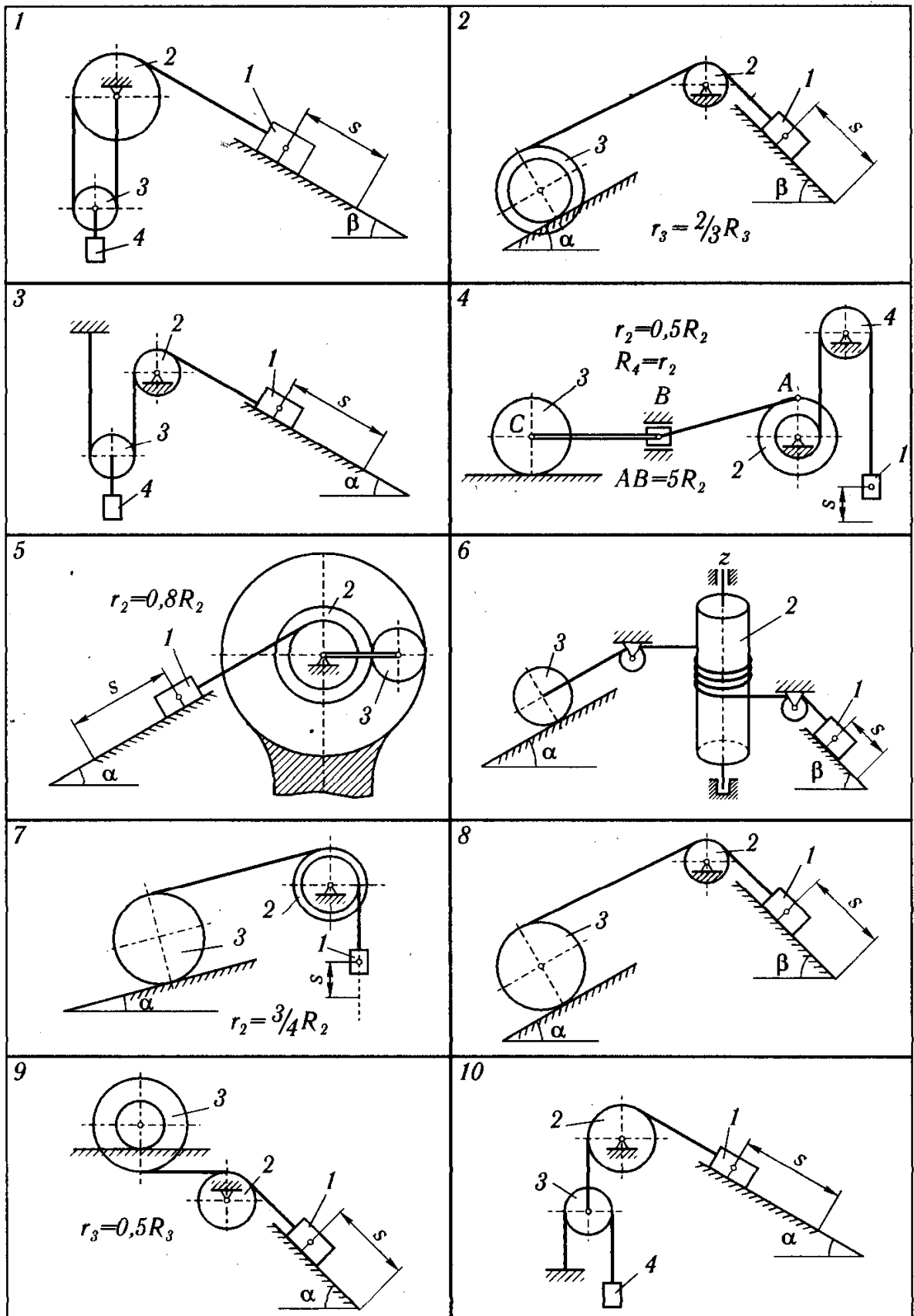


Fig. 106

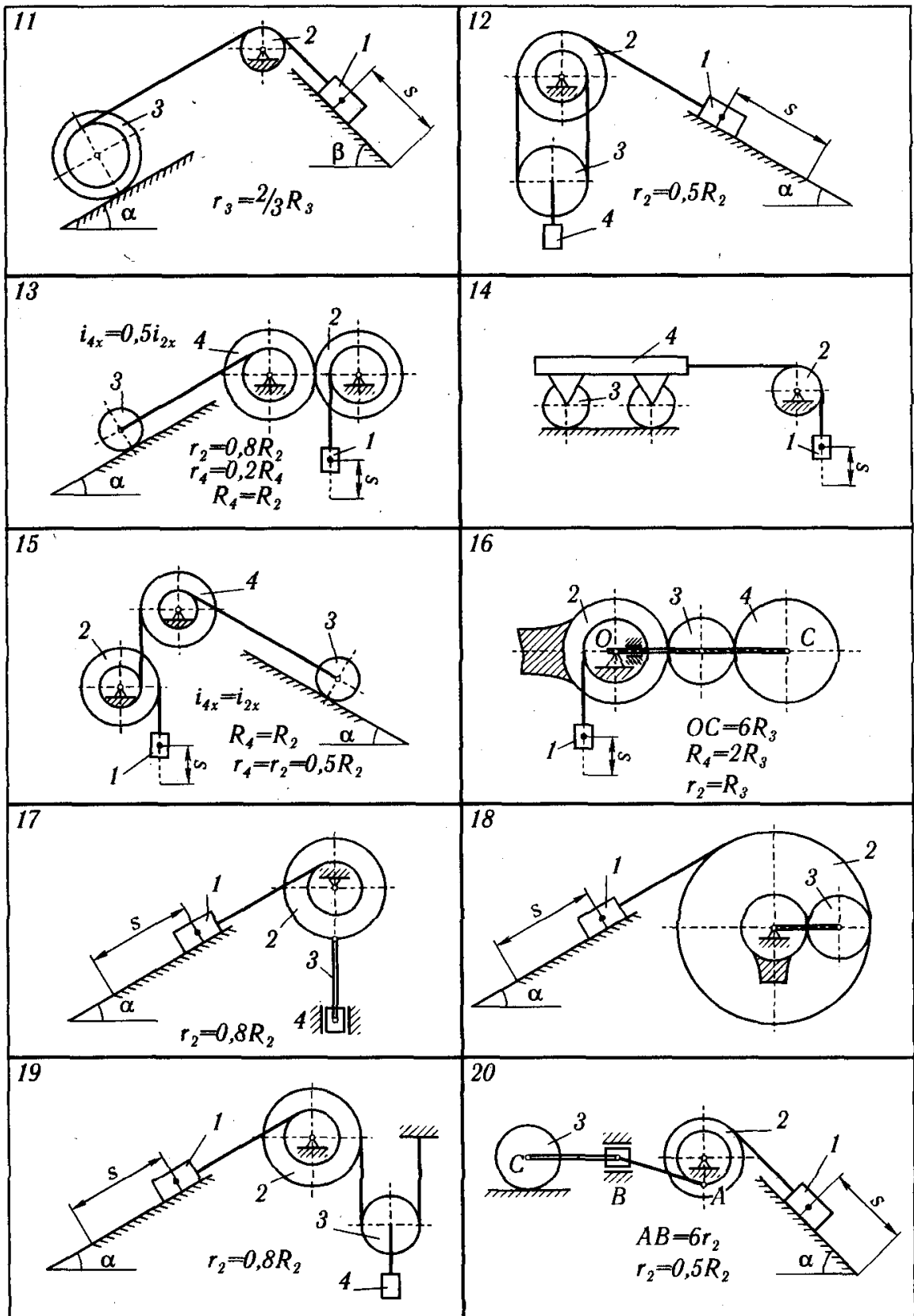


Fig. 107



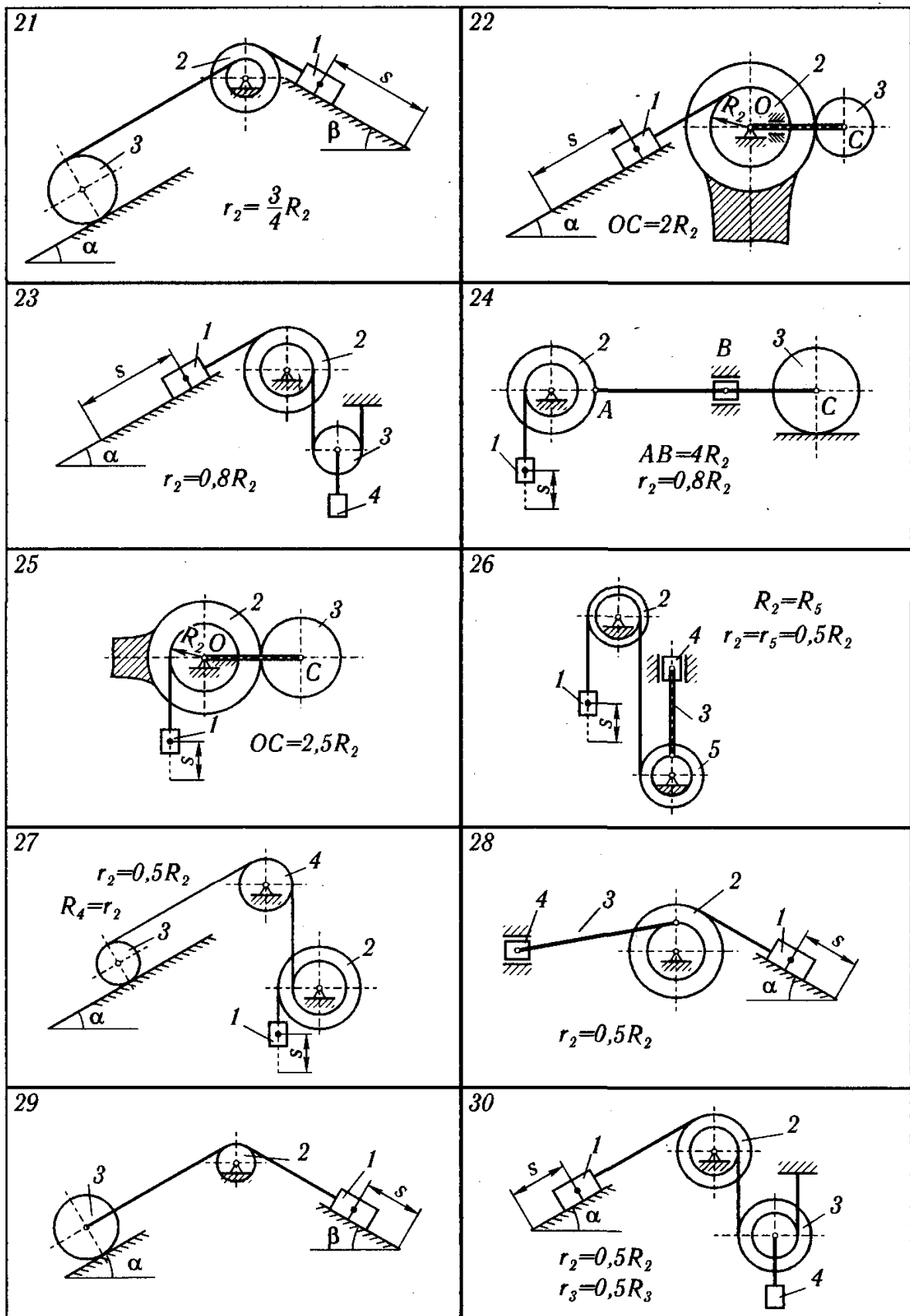


Fig. 108

Tab. 1

Variant number (Fig.106-108)	$m_1$	$m_2$	$m_3$	$m_4$	$R_2$	$R_3$	$i_2$	$i_3$	$\alpha$	$\beta$	$f$	$\delta, cm$	$s, m$	Notes
	kg													
1	$m$	$4m$	$1/5m$	$4/3m$	–	–	–	–	–	60	0,10	–	2	
2	$m$	$1/2m$	$1/3m$	–	–	30	–	20	30	45	0,22	0,20	2	
3	$m$	$m$	$1/10m$	$m$	–	–	–	–	45	–	0,10	–	2	
4	$m$	$2m$	$40m$	$m$	20	40	18	–	–	–	–	0,30	$0,1\pi$	Neglect the masses of the slider $B$ , links $AB$ and $BC$
5	$m$	$2m$	$m$	–	20	15	18	–	60	–	0,12	–	$0,28\pi$	Neglect the mass of the cage
6	$m$	$3m$	$m$	–	–	28	–	–	30	45	0,10	0,28	1,5	
7	$m$	$2m$	$2m$	–	16	25	14	–	30	–	–	0,20	2	
8	$m$	$1/2m$	$1/3m$	–	–	30	–	–	30	45	0,15	0,20	1,75	
9	$m$	$2m$	$9m$	–	–	30	–	20	30	–	0,12	0,25	1,5	
10	$m$	$1/4m$	$1/4m$	$1/5m$	–	–	–	–	60	–	0,10	–	3	
11	$m$	$1/2m$	$1/4m$	–	–	30	–	25	30	45	0,17	0,20	2,5	
12	$m$	$1/2m$	$1/5m$	$m$	30	–	20	–	30	–	0,20	–	2,5	
13	$m$	$2m$	$5m$	$2m$	30	20	26	–	30	–	–	0,24	2	
14	$m$	$1/2m$	$5m$	$4m$	–	25	–	–	–	–	–	0,20	2	The masses of all the wheels are equal
15	$m$	$1/2m$	$4m$	$1/2m$	20	15	18	–	60	–	–	0,25	1,5	

Continuation of the tab. 1

Variant number (Fig.106-108)	$m_1$	$m_2$	$m_3$	$m_4$	$R_2$	$R_3$	$i_2$	$i_3$	$\alpha$	$\beta$	$f$	$\delta, cm$	$s, m$	Notes
	$kg$													
16	$m$	$1/10m$	$1/20m$	$1/10m$	10	12	–	–	–	–	–	–	$0,05\pi$	Neglect the mass of the cage
17	$m$	$1/4m$	$1/5m$	$1/10m$	20	–	15	–	60	–	0,10	–	$0,16\pi$	Consider connecting rod 3 as a thin homogeneous rod
18	$m$	$3m$	$m$	–	35	15	32	–	60	–	0,15	–	$0,2\pi$	Neglect the mass of the cage
19	$m$	$1/3m$	$1/10m$	$m$	24	–	20	–	60	–	0,15	–	1,5	
20	$m$	$2m$	$20m$	–	20	15	16	–	30	–	0,10	0,20	$0,2\pi$	Neglect the masses of the slider $B$ , links $AB$ and $BC$
21	$m$	$m$	$2m$	–	20	20	16	–	30	45	0,20	0,32	1,2	
22	$m$	$1/2m$	$1/4m$	–	20	10	–	–	60	–	0,17	–	$0,1\pi$	Neglect the mass of the cage
23	$m$	$m$	$1/10m$	$4/5m$	20	–	18	–	30	–	0,10	–	1	
24	$m$	$3m$	$20m$	–	20	30	18	–	–	–	–	0,60	$0,08\pi$	Neglect the masses of the slider $B$ , links $AB$ and $BC$
25	$m$	$1/3m$	$1/4m$	–	16	20	–	–	–	–	–	–	$0,04\pi$	Neglect the mass of the cage
26	$m$	$1/2m$	$m$	$1/3m$	30	–	20	–	–	–	–	–	$0,6\pi$	The masses and moments of inertia of pulleys 2 and 5 are equal. Consider connecting rod 3 as a thin homogeneous rod
27	$m$	$m$	$6m$	$1/2m$	20	20	16	–	30	–	–	0,20	2	
28	$m$	$2m$	$3m$	–	20	–	14	–	60	–	0,10	–	$0,1\pi$	Consider connecting rod 3 as a thin homogeneous rod
29	$m$	$1/4m$	$1/8m$	–	–	35	–	–	15	30	0,20	0,20	2,4	
30	$m$	$1/2m$	$3/10m$	$3/2m$	26	20	20	18	30	–	0,12	–	2	

### 8.3. Application of Virtual Work Principle to the Static Problems

Mechanisms are in a state of equilibrium. Their schemas are shown in Fig. 111-113. The necessary data are given in tab. 2. By making use of virtual work principle determine quantity specified in table 2. Neglect the forces of resistance. Mechanisms are located in a vertical plane for variants 3, 6, 10, 14, 16, 18, 19, 25 and 30, the rest of mechanisms are in a horizontal plane.

*Example.* Given are:  $Q = 100 \text{ N}$ ;  $c = 5 \text{ N/cm}$ ;  $r_1 = 20 \text{ cm}$ ;  $r_2 = 40 \text{ cm}$ ;  $r_3 = 10 \text{ cm}$ ;  $OA = l = 50 \text{ cm}$ ;  $\alpha = 30^\circ$ ;  $\beta = 90^\circ$  (Fig. 109)

Determine deformation of the spring  $h$  in a state of equilibrium neglecting the weight of the links  $OA$  and  $AB$ .

*Solution.* Mechanism is under the action of the following balanced force system: elastic force  $F$ ,  $G_1$  – weight of the shaft  $l$  with gear 2,  $G_3$  – weight of the gear 3,  $G_4$  – weight of the slider  $B$ ,  $Q$  – weight of the load, and the reactions of constraints (supports) which are not shown in Fig. 109.

Make up equation of virtual work principle taking into account that all constraints are ideal (6.6):

$$\sum \delta A_k^a = 0.$$

There are following virtual displacements consistent with the constraints of mechanism in this problem: rotation of the shaft  $l$  with the gear 2 on angle  $\delta\varphi_1$ , rotation of the gear 3 on angle  $\delta\varphi_3$  and vertical translation of the load  $\delta s_Q$ . The slider  $B$  has a virtual displacement  $\delta s_B$  (along piston rod guide), and point  $A$  has a displacement  $\delta s_A$  ( $\delta s_A$  is perpendicular to  $OA$ ). Equation of the virtual work principle has a form:

$$Q\delta s_Q - F\delta s_B = 0.$$

Define virtual displacements relation. Since the load  $Q$  is fasten to inextensible string and there is no sliding between the string and the shaft, the displacement of the load  $Q$  equals the displacement of the points of the rim of the rod  $l$ . Therefore, the angle of rotation of the shaft with gear 2 is

$$\delta\varphi_1 = \delta s_Q / r_1.$$

The displacement of the point  $K$  is

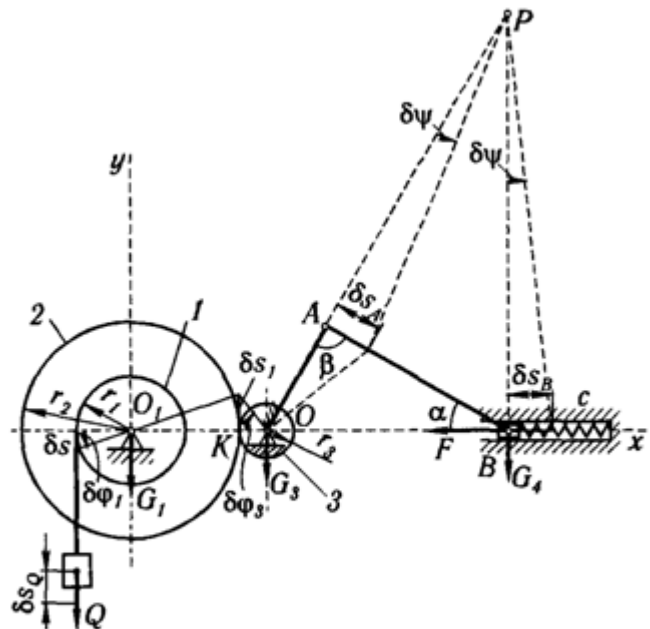


Fig. 109

$$\delta s_1 = r_2 \delta \varphi_1 = (r_2/r_1) \delta s_Q.$$

Since there is no sliding between the rods 2 and 3, the virtual displacements of their points of contact are equal and the angle of rotation of the gear 3 is

$$\delta \varphi_3 = \delta s_1/r_3 = [r_2/(r_1 r_3)] \delta s_Q.$$

The crankshaft  $OA$  is rigidly connected with the gear 3 and so

$$\delta s_A = OA \delta \varphi_3 = [r_2 l / (r_1 r_3)] \delta s_Q.$$

In order to determine the dependency between virtual displacements  $\delta s_B$  and  $\delta s_A$ , find the position of the instantaneous rotation center of the link  $AB$ , i.e., point  $P$ .

Then

$$\delta s_B / \delta s_A = PB/PA; \quad \delta s_B = (PB/PA) \delta s_A.$$

From the  $\triangle APB$

$$PB/PA = 1/\cos 30^\circ.$$

Hence,

$$\delta s_B = [r_2 l / (r_1 r_3 \cos 30^\circ)] \delta s_Q.$$

Elastic force of the spring is proportional to its deformation:  $F = ch$ . Then from equation of the virtual work principle we have

$$Q \delta s_Q - ch [r_2 l / (r_1 r_3 \cos 30^\circ)] \delta s_Q = 0,$$

whence

$$h = \frac{Q r_1 r_3 \cos 30^\circ}{c r_2 l}; \quad h = 1,74 \text{ cm}.$$

Consequently the spring is compressed on 1,74 cm.

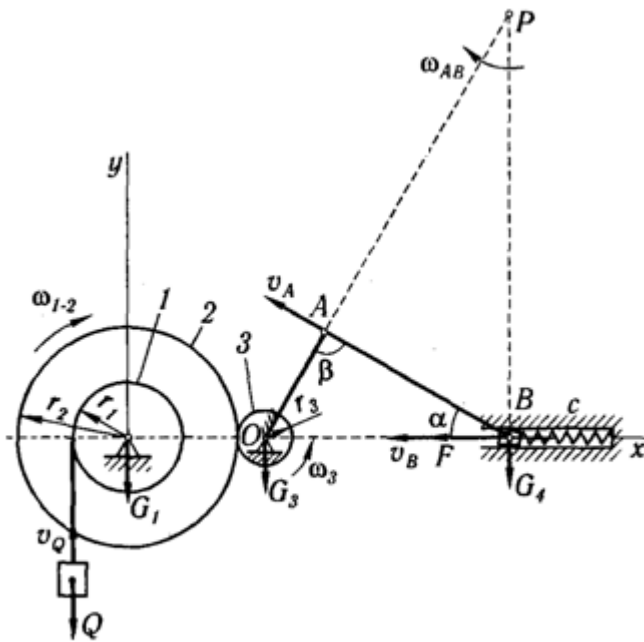


Fig.110

Let us resolve this problem by composing equation of power which expresses virtual velocities principle:

$$\sum \mathbf{P}_i \cdot \mathbf{v}_i = 0 \text{ or } \sum P_i v_i \cos(\widehat{\mathbf{P}_i, \mathbf{v}_i}) = 0,$$

where  $\mathbf{P}_i$  are all the active forces applied to a system,  $\mathbf{v}_i$  are velocities of their points of application.

Give the shaft 1 with the gear 2 a virtual angular velocity  $\omega_{1-2}$  around its axis of rotation, let it be clockwise (Fig.110). Then the load  $Q$  gets vertical velocity  $v_Q$ . The gear 3 with rigidly connected crankshaft  $OA$  acquires angular velocity  $\omega_3$  around  $O$ .

The link  $AB$  will have angular velocity  $\omega_{AB}$  which can be represented around instantaneous velocity center  $P$ . This center is located at point of intersection of perpendiculars erected to velocities  $\mathbf{v}_A$

and  $v_B$  (velocity  $v_A$  belonging to the crankshaft  $OA$  is perpendicular to  $OA$ , and velocity  $v_B$  belonging to the slider is parallel to the piston rod guide).

Compose equation of virtual velocity principle:

$$-Qv_Q + Fv_B = 0, \text{ or } -Qv_Q + chv_B = 0.$$

There are three unknown quantities here: deformation of the spring  $h$ , velocities  $v_Q$  and  $v_B$ .

Velocity of the load equals velocities of the points of the rim since the string is inextensible, and so

$$v_Q = \omega_{1-2}r_1.$$

Velocities of the point of contact  $K$  of the gears 2 and 3 are

$$v_K = \omega_{1-2}r_2, \quad v_K = \omega_3r_3.$$

They are equal since there is no sliding between the gears.

As far as a point  $A$  belongs simultaneously to the crankshaft  $OA$  and to the link  $AB$ , we have

$$v_A = \omega_3 \cdot OA; \quad v_A = \omega_{AB} \cdot AP.$$

Velocity of the point  $B$  of the link  $AB$  is

$$v_B = \omega_{AB} \cdot BP.$$

Therefore,

$$\omega_{1-2}r_2 = \omega_3r_3; \quad \omega_3 \cdot OA = \omega_{AB} \cdot AP,$$

whence

$$\omega_3 = \frac{\omega_{1-2}r_2}{r_3}; \quad \omega_{AB} = \frac{\omega_3 \cdot OA}{AP} = \frac{\omega_{1-2}r_2 \cdot OA}{r_3 \cdot AP}.$$

Then

$$v_B = \omega_{AB} \cdot BP = \frac{\omega_{1-2}r_2 \cdot OA \cdot BP}{r_3 \cdot AP}.$$

From the  $\Delta APB$

$$AP = BP \cdot \cos 30^\circ.$$

So

$$v_B = \frac{\omega_{1-2}r_2 \cdot OA}{r_3 \cos 30^\circ}.$$

Thus, equation of the virtual velocity principle acquires the following form:

$$-Q\omega_{1-2}r_1 + ch \frac{\omega_{1-2}r_2 l}{r_3 \cos 30^\circ} = 0.$$

Dividing this equation by  $\omega_{1-2}$ , we find deformation of the spring

$$h = \frac{Qr_1 r_3 \cos 30^\circ}{cr_2 l}.$$

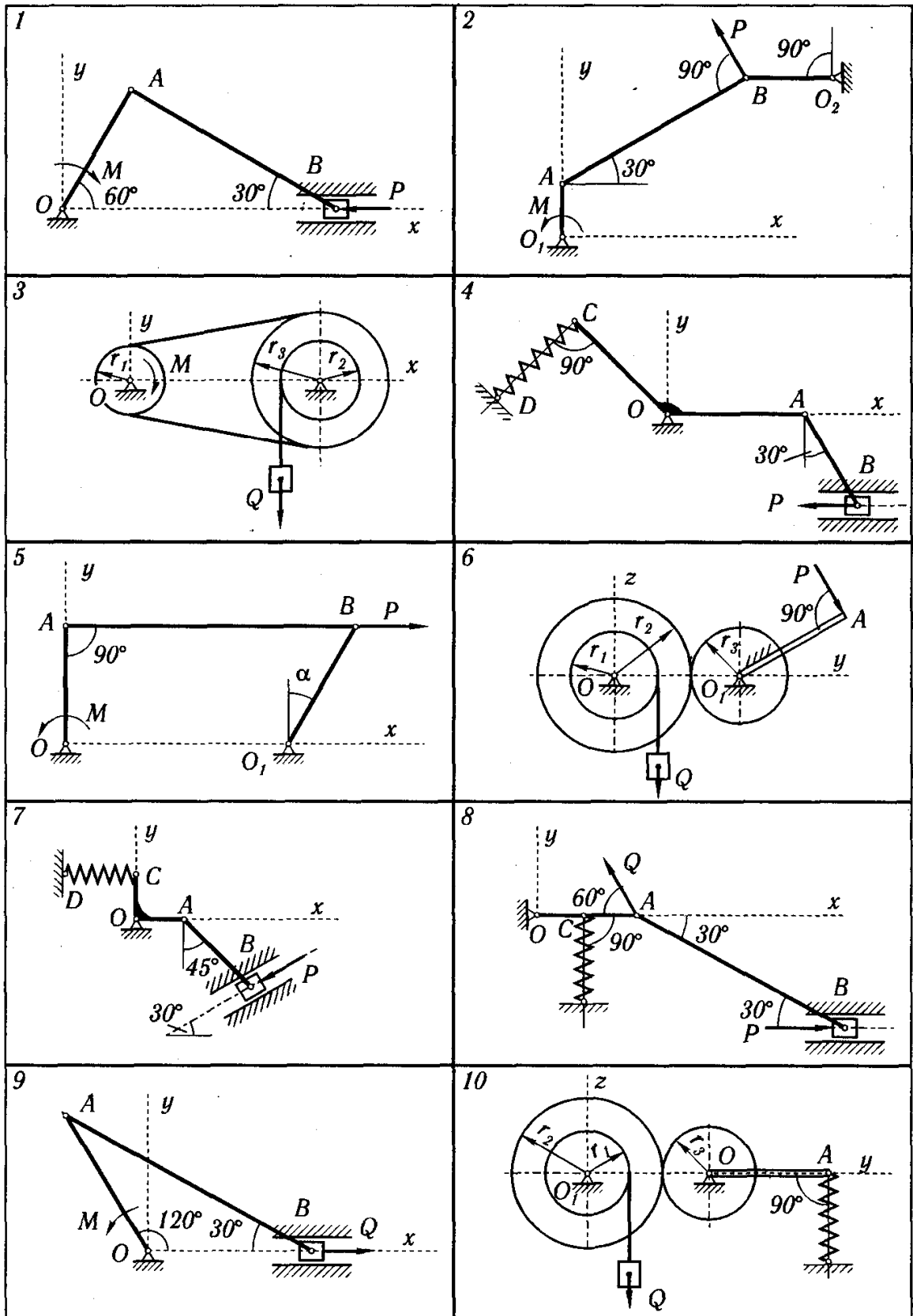


Fig. 111

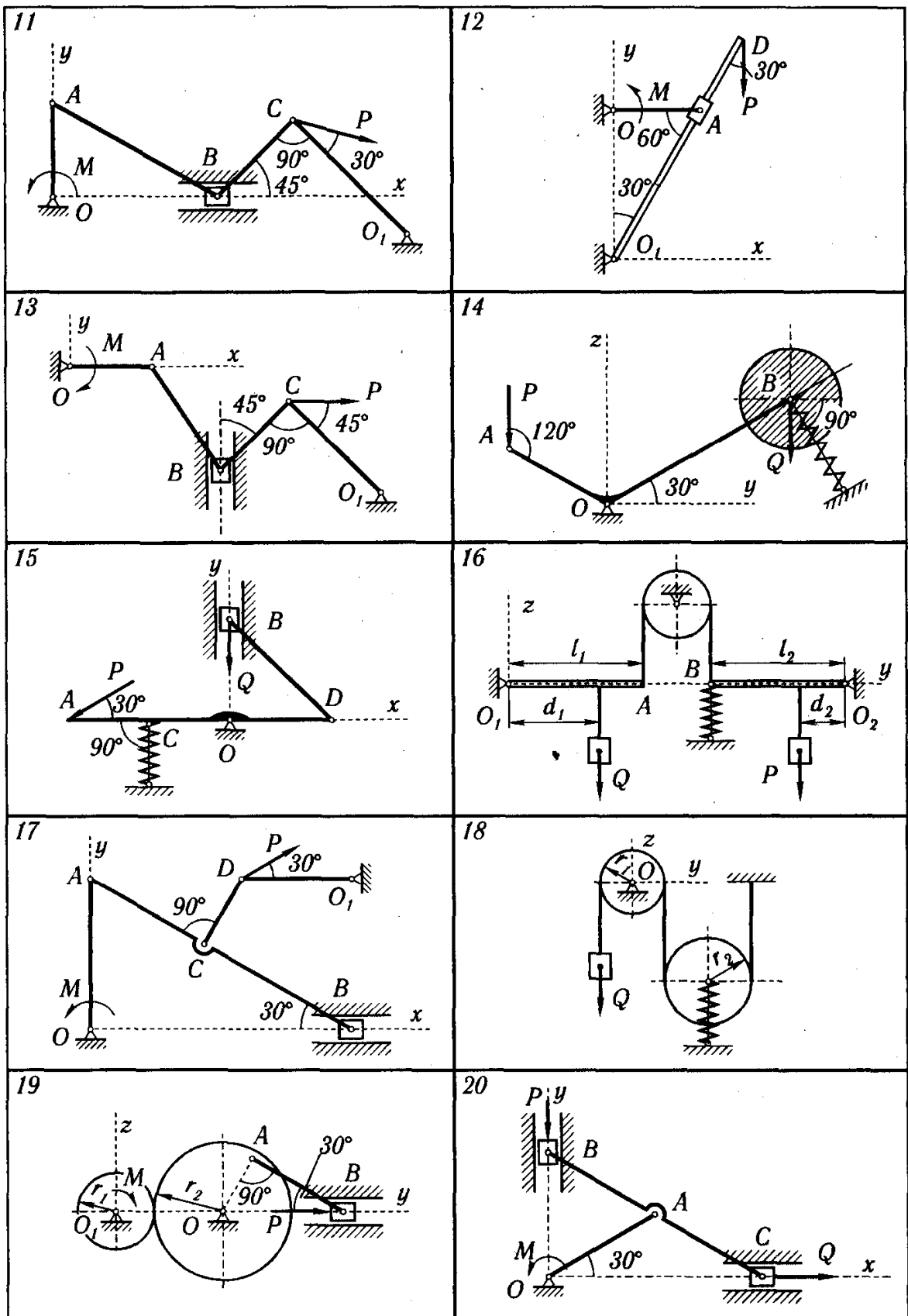


Fig. 112



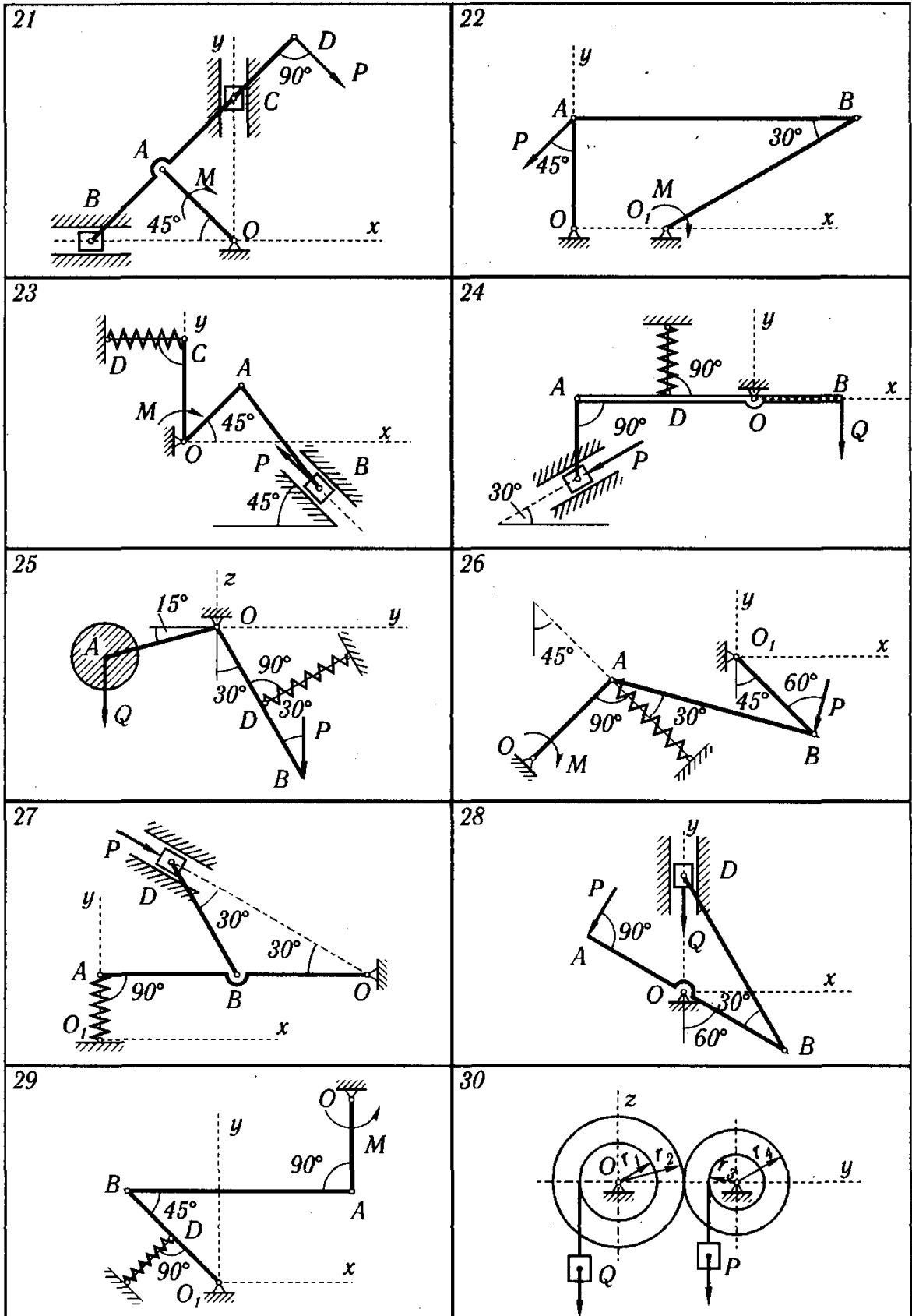


Fig. 113

Tab. 2

Variant number (Fig.111-113 -)	Linear dimensions	Forces, $N$		Moment of a couple $M, N \cdot m$	Stiffness factor $c, N/cm$	Spring deflection $h, cm$	Determine	Notes
		$Q$	$P$					
1	$OA = 10 \text{ cm}$	–	–	20	–	–	$P$	
2	$O_1A = 20 \text{ cm}$	–	100	–	–	–	$M$	
3	$r_1 = 20 \text{ cm}, r_2 = 30 \text{ cm},$ $r_3 = 40 \text{ cm}$	–	–	100	–	–	$Q$	
4	$OC : OA = 4 : 5$	–	200	–	–	4	$c$	
5	$OA = 100 \text{ cm}$	–	–	10	–	–	$P$	
6	$r_1 = 15 \text{ cm}, r_2 = 50 \text{ cm},$ $r_3 = 20 \text{ cm}, O_1A = 80 \text{ cm}$	200	–	–	–	–	$P$	Neglect the weight of the handle $O_1A$
7	$OC = OA$	–	–	–	10	3	$P$	Spring is compressed
8	$OC = AC$	–	200	–	10	2	$Q$	The same
9	$OA = 20 \text{ cm}$	200	–	–	–	–	$M$	
10	$r_1 = 15 \text{ cm}, r_2 = 40 \text{ cm},$ $r_3 = 20 \text{ cm}, OA$ $= 100 \text{ cm}$	$2 \cdot 10^3$	–	–	–	4	$c$	Neglect the weight of the handle $OA$
11	$OA = 20 \text{ cm}$	–	–	300	–	–	$P$	
12	$O_1D = 60 \text{ cm}, OA$ $= 20 \text{ cm}$	–	–	100	–	–	$P$	
13	$OA = 40 \text{ cm}$	–	–	200	–	–	$P$	
14	$OB = 2 \cdot OA$	20	–	–	25	3	$P$	Neglect the weight of $OA$ and $OB$ ; spring is stretched
15	$AC = OC = OD$	$3 \cdot 10^3$	–	–	250	3	$P$	Spring is compressed

Variant number (Fig.111-113)	Linear dimensions	Forces, $N$		Moment of a couple $M, N \cdot m$	Stiffness factor $c, N/cm$	Spring deflection $h, cm$	Determine	Notes
		$Q$	$P$					
16	$d_1 = 80 \text{ cm}, d_2 = 25 \text{ cm},$ $l_1 = 100 \text{ cm}, l_2 = 50 \text{ cm}$	$5 \cdot 10^3$	—	—	100	4	$P$	Neglect the weight of $O_1A$ and $O_2B$ . Spring is compressed
17	$OA = 20 \text{ cm}$	—	—	200	—	—	$P$	
18	$r_1 = 20 \text{ cm},$ $r_2 = 30 \text{ cm}$	200	200	—	100	—	$h$	$P$ — weight of the block of the radius $r_2$
19	$r_1 = 20 \text{ cm},$ $r_2 = 30 \text{ cm}, OA = 25 \text{ cm}$	—	—	100	—	—	$P$	Neglect the weight of the link $AB$
20	$OA = AB = AC = 50 \text{ cm}$	50	100	—	—	—	$M$	
21	$OA = AB = AC = DC =$ $= 25 \text{ cm}$	—	200	—	—	—	$M$	
22	$OA = 40 \text{ cm}$	—	—	400	—	—	$P$	
23	$OC = 2OA = 100 \text{ cm}$	—	200	50	50	—	$h$	
24	$AD = OD = OB \text{ cm}$	—	250	—	150	2,5	$Q$	Spring is compressed
25	$OD = DB = 0,8AO$	400	—	—	120	3	$P$	Neglect the weight of $AO$ and $BO$ . Spring is stretched
26	$OA = 25 \text{ cm}$	—	500	120	—	2	$c$	Spring is stretched
27	$OB = AB$	—	—	—	180	2	$P$	
28	$OB = (5/4)OA$	—	450	—	—	—	$Q$	
29	$AO = 30 \text{ cm}, BD = O_1D$	—	—	120	100	—	$h$	
30	$r_1 = 15 \text{ cm}, r_2 = 36 \text{ cm},$ $r_3 = 10 \text{ cm}, r_4 = 20 \text{ cm}$	—	600	—	—	—	$Q$	

### 8.4. Application of General Equation of Dynamics to Study of Motion of Mechanical System with One Degree of Freedom

For the given mechanical system define accelerations of weights and a tension in threads to which weights are attached. Neglect mass of threads, friction of a rolling and force of resistance in bearings. The system moves from a state of rest.

Variants of mechanical systems are shown in Fig. 116-118, and the necessary data are represented in tab.3. Radii of gyration are given with respect to central axes perpendicular to the plane of figure.

Assume coefficient of friction identical both for body's sliding along the plane and for braking shoe (variants 9—12).

Assume pulleys and blocks for which the radii of gyration are not given as solid homogeneous disks.

*Example.* It is given:  $G_1 = G_2 = 2G$ ;  $G_3 = G_4 = G$ ;  $R = 2r$ ;  $i_{2x} = r\sqrt{2}$ ;  $f = 0,2$ .

The block 3 is solid homogeneous cylinder (Fig. 114). Define accelerations of weights 1 and 4 and tension of branches of a thread 1—2 and 3—4.

*Solution.* Let us apply to the problem solution the general equation of dynamics. As the system starts moving from a state of rest, directions of accelerations of bodies correspond to directions of their motion.

Whereas among the forces acting on bodies of system, there is a force of a friction, it is expedient to find a true direction of motion according to initial data in order to show correct direction of force of friction.

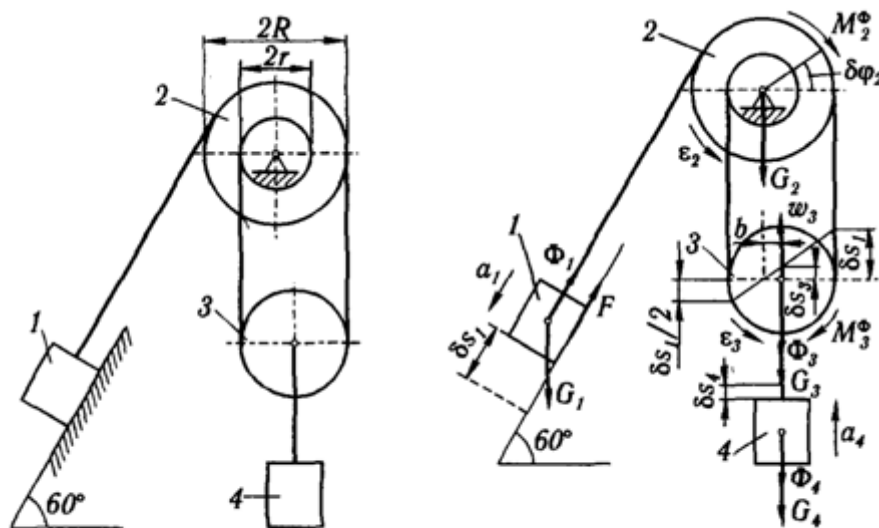


Fig. 114

If the direction of motion of system is chosen wrongly, required acceleration turns out with a sign «—». In this case it is necessary to change directions of force of friction and forces of inertia and to make corresponding corrections in the general equation of dynamics.

In the given example motion of a system is that weight 1 falls.

Let's show force system: a gravity  $G_1$ — load 1,  $G_2$ — the block 2,  $G_3$ — the

block 3 and  $G_4$ — load 4, and also  $F$ — force of a sliding friction of load 1 on an inclined plane (Fig. 114).

Let's apply forces of inertia. Force of inertia of load 1 making translational motion with acceleration  $\mathbf{a}_1$  is expressed by a vector

$$\Phi_1 = -m_1 \mathbf{a}_1.$$

Inertial forces of the block 2 rotating around fixed axis with angular acceleration  $\varepsilon_2$  are reduced to a couple. Its moment is

$$M_2^\Phi = J_{2x} \varepsilon_2.$$

Inertial forces of the block 3 making a plane motion are reduced to a force

$$\Phi_3 = -m_3 \mathbf{a}_3,$$

where  $\mathbf{a}_3$  — acceleration of the center of mass of the block 3, and to a couple of forces, which moment is

$$M_3^\Phi = J_{3x} \varepsilon_3,$$

where  $\varepsilon_3$  — angular acceleration of the block.

Inertial force of a load 4 making translational motion with acceleration  $\mathbf{a}_4$  is

$$\Phi_4 = -m_4 \mathbf{a}_4.$$

Let's give the system a virtual displacement to a direction of its true motion (Fig. 114) (it is possible to give virtual displacement in the opposite direction).

Writing the general equation of dynamics, we obtain

$$G_1 \delta s_1 \sin 60^\circ - F \delta s_1 - \Phi_1 \delta s_1 - M_2^\Phi \delta \varphi_2 - G_3 \delta s_3 - \Phi_3 \delta s_3 - M_3^\Phi \delta \varphi_3 - G_4 \delta s_4 - \Phi_4 \delta s_4 = 0,$$

where  $\delta \varphi_2$  and  $\delta \varphi_3$ — angles of rotation of blocks 2 and 3.

Relations between virtual displacements are the same, as for relations between corresponding velocities.

Let's express velocities of the centers of mass and angular velocities of bodies of system as a function of velocity of a body 1.

As is shown in Fig. 114, the instantaneous center of zero velocity of the block 3 is on one vertical with the center of the block 2. Distance between the instantaneous center of zero velocity and the center of the block 3 is

$$b = \frac{3r}{2} - r = \frac{r}{2}.$$

Now we find

$$\left. \begin{aligned} \omega_2 = \omega_3 = v_1/R = v_1/2r; \\ v_3 = v_4 = \omega_3 b = v_1/4. \end{aligned} \right\}$$

The same dependences are between virtual displacements

$$\left. \begin{aligned} \delta \varphi_2 = \delta \varphi_3 = \delta s_1/(2r); \\ \delta s_3 = \delta s_4 = \delta s_1/4. \end{aligned} \right\}$$

General equation taking into account these formulas becomes

$$G_1 \sin 60^\circ - F - \Phi_1 - M_2^\Phi/(2r) - G_3/4 - \Phi_3/4 - M_3^\Phi/(2r) - G_4/4 - \Phi_4/4 = 0.$$

The same equation can be obtained, if to work out the equation of power, having gave system virtual velocities. The relationships resulted for the real velocities of bodies of system are the same, as for any virtual velocities.

Considering that

$$G_1 = G_2 = 2G = 2mg; \quad G_3 = G_4 = G = mg,$$

we have

$$\left\{ \begin{array}{l} F = fG \cos 60^\circ = fmg; \\ \Phi_1 = m_1 a_1 = 2ma_1; \\ M_2^\Phi = J_{2x} \varepsilon_2 = m_2 i_{2x}^2 \varepsilon_2 = 4mr^2 \varepsilon_2; \\ \Phi_3 = m_3 a_3 = ma_3; \\ M_3^\Phi = J_{3x} \varepsilon_3 = [m_3 (1,5r)^2 / 2] \varepsilon_3 = 9mr^2 \varepsilon_3 / 8; \\ \Phi_4 = m_4 a_4 = ma_4. \end{array} \right.$$

Dependences between accelerations are

$$\left. \begin{array}{l} \varepsilon_2 = \varepsilon_3 = a_1 / (2r); \\ a_3 = a_4 = a_1 / 4. \end{array} \right\}$$

Then we will obtain

$$g\sqrt{3} - fg - 2a_1 - a_1 - g/4 - a_1/16 - 9a_1/32 - g/4 - a_1/16 = 0,$$

whence

$$a_1 = \frac{g(\sqrt{3} - f - 0,5)}{3,41}; \quad a_1 = 2,96 \text{ m/sec}^2;$$

$$a_4 = \frac{a_1}{4}; \quad a_4 = 0,74 \text{ m/sec}^2.$$

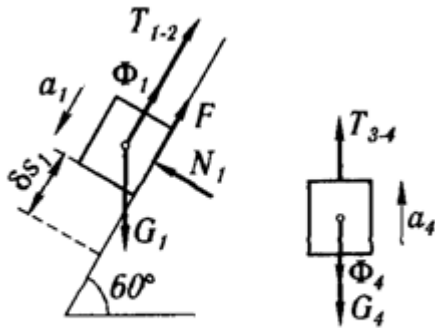


Fig. 115

In order to define tension in a thread 1—2, we will mentally cut this thread and replace its action on a load 2 by reaction  $T_{1-2}$  (Fig. 115).

Then general equation of dynamics will be  
 $G_1 \delta s_1 \sin 60^\circ - F \delta s_1 - \Phi_1 \delta s_1 - T_{1-2} \delta s_1 = 0,$   
whence  $T_{1-2} = G_1 \sin 60^\circ - F - \Phi_1 =$   
 $2G \sin 60^\circ - 2Gf \cos 60^\circ - 2(G/g)a_1; T_{1-2} =$

0,93G.

In order to define tension in a thread 3—4, we will mentally cut this thread and replace its action on a load 4 by reaction  $T_{3-4}$  (Fig. 115).

Without making the general equation of dynamics, on the basis of D'Alembert's principle we have

$$T_{3-4} = G_4 + \Phi_4 = G + (G/g)a_4; \quad T_{3-4} = 1.08G.$$

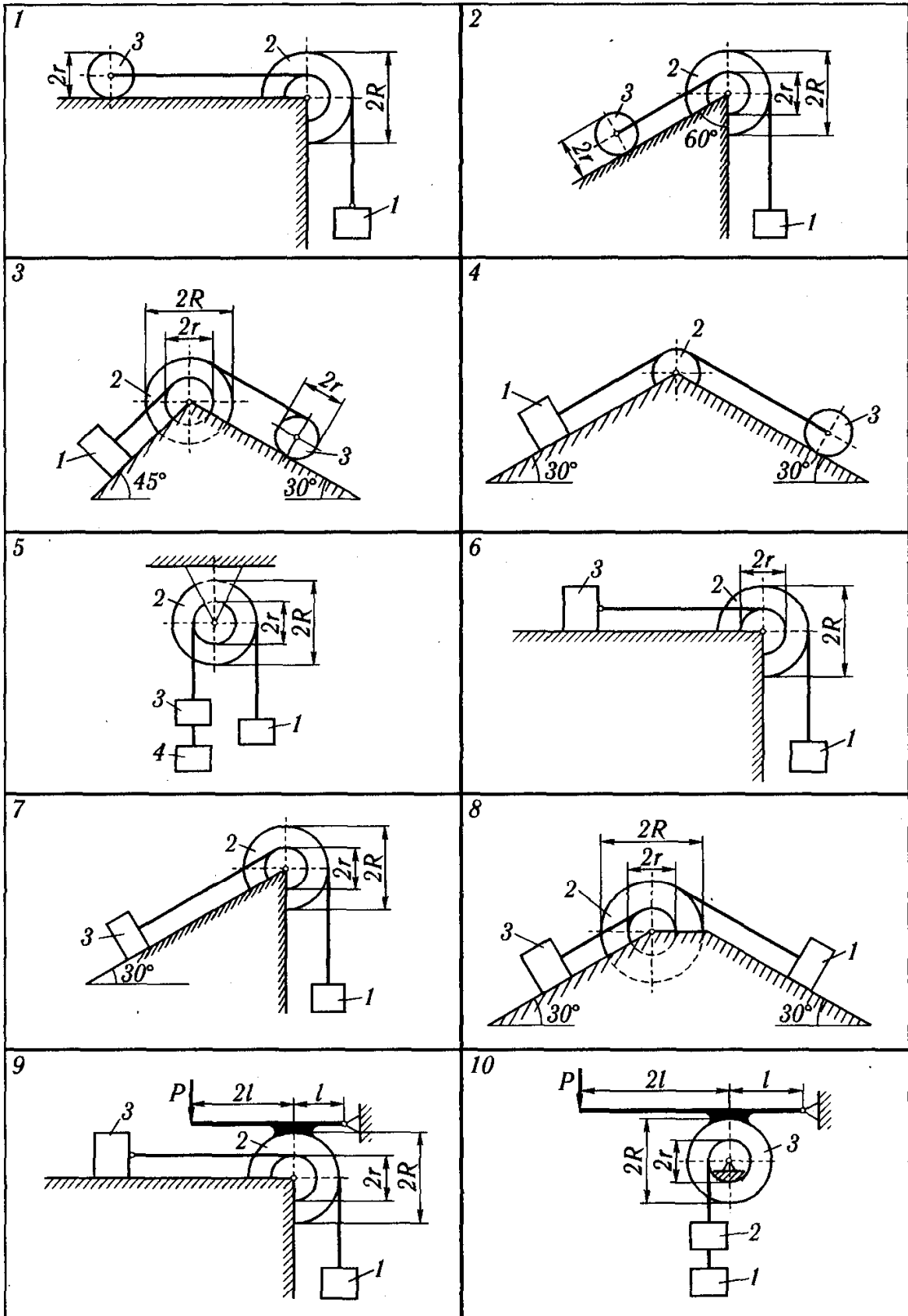


Fig. 116

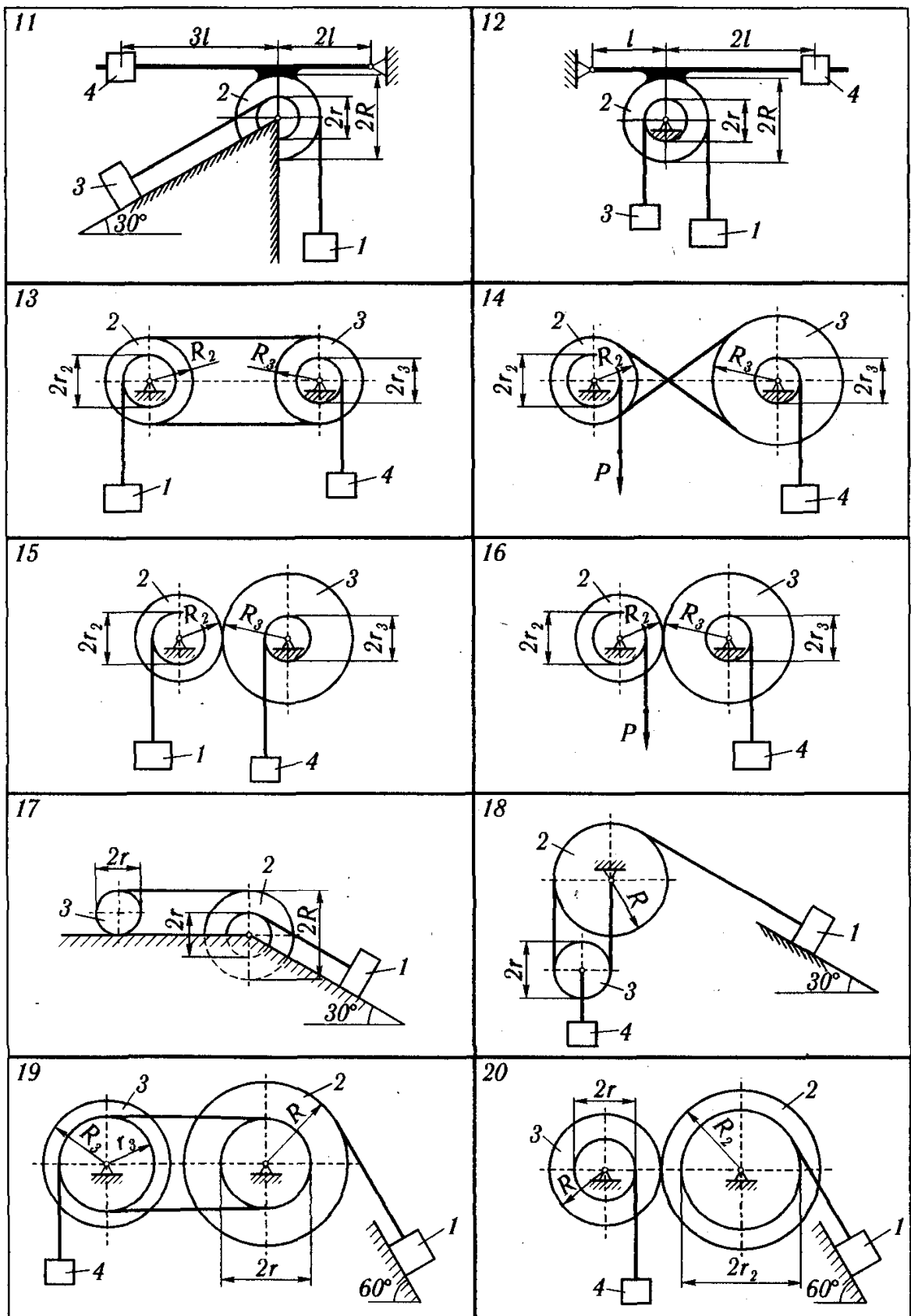


Fig. 117



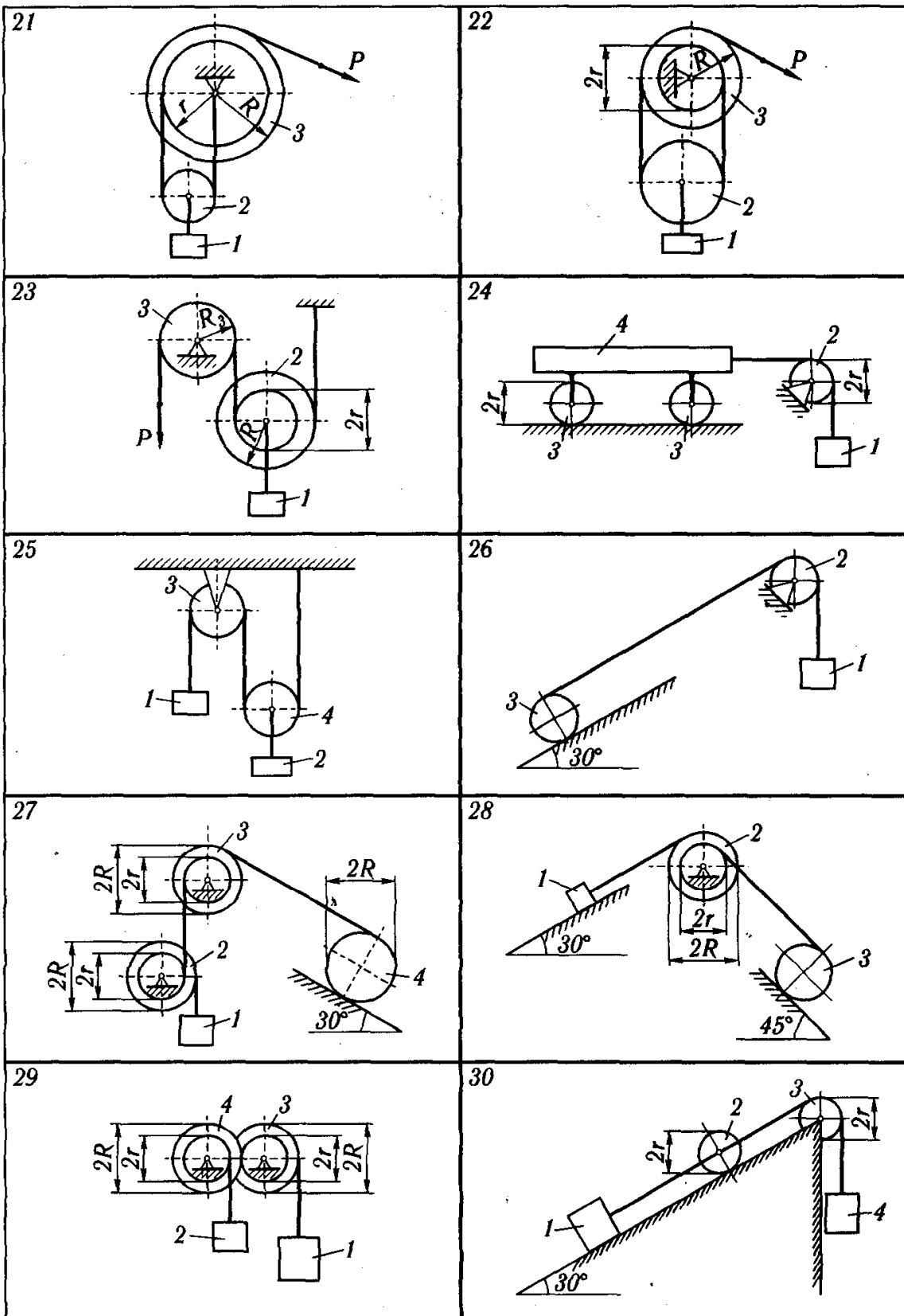


Fig. 118

Tab. 3

Variant number (Fig.116-118)	Gravity force				$\frac{R}{r}$	Radii of gyration		$P$	Coefficient of friction $f$	Supplementary data
	$G_1$	$G_2$	$G_3$	$G_4$		$i_{2x}$	$i_{3x}$			
1	$G$	$G$	$3G$	—	2	$r\sqrt{2}$	—	—	—	
2	$G$	$G$	$G$	—	2	$r\sqrt{2}$	—	—	—	
3	$3G$	$G$	$G$	—	2	$r\sqrt{2}$	—	—	0,1	
4	$G$	$G$	$2G$	—	—	—	—	—	0,2	$r_2 = r_3$
5	$2G$	$G$	$G$	$G$	3	$2r$	—	—	—	
6	$2G$	$G$	$2G$	—	3	$2r$	—	—	0,2	
7	$2G$	$G$	$2G$	—	3	$2r$	—	—	0,2	
8	$2G$	$G$	$2G$	—	3	$2r$	—	—	0,2	
9	$2G$	$G$	$2G$	—	3	$2r$	—	$0,2G$	0,2	
10	$2G$	$2G$	$G$	—	4	—	$2r$	$G/3$	0,4	
11	$2G$	$G$	$2G$	$0,2G$	3	$2r$	—	—	0,2	
12	$2G$	$G$	$2G$	$0,2G$	3	$2r$	—	—	0,2	
13	$4G$	$2G$	$G$	$4G$	—	$r_2\sqrt{2}$	$2r_3$	—	—	$r_2 = 2r_3; R_2 = R_3$
14	—	$G$	$G$	$4G$	—	$r_2\sqrt{2}$	$2r_3$	$8G$	—	$r_2 = 2r_3; R_3 = 1,5R_2$
15	$4G$	$G$	$2G$	$4G$	—	$r_2\sqrt{2}$	$2r_3$	—	—	$r_2 = 2r_3; R_3 = 1,5R_2$
16	—	$G$	$2G$	$4G$	—	$r_2\sqrt{2}$	$2r_3$	$4G$	—	$r_2 = 2r_3; R_3 = 1,5R_2$
17	$2G$	$G$	$G$	—	2	$r\sqrt{2}$	—	—	0,1	
18	$3G$	$0,2G$	$0,1G$	$0,5G$	2	—	—	—	0,4	
19	$4G$	$0,3G$	$0,2G$	$3G$	3	$2r$	$1,2r$	—	0,1	$r_3 = 1,2r; R_3 = 1,2r_3$
20	$4G$	$0,2G$	$0,1G$	$3G$	2	$1,6r$	$r\sqrt{2}$	—	0,2	$r_2 = 1,5r; R_2 = 1,2r_2$
21	$5G$	$0,1G$	$0,2G$	—	3	—	$r\sqrt{2}$	$G$	—	
22	$G$	$0,2G$	$0,3G$	—	2	—	$r\sqrt{2}$	$G$	—	
23	$G$	$0,2G$	$0,1G$	—	1,5	$1,2r$	—	$2G$	—	$R_3 = 1,2r$
24	$2G$	$G$	$G$	$8G$	—	—	—	—	—	Masses of the wheels are equal
25	$6G$	$2G$	$2G$	$G$	—	—	—	—	—	$r_3 = r_4$
26	$6G$	$G$	$2G$	—	—	—	—	—	—	$r_3 = r_2$
27	$G$	$G$	$G$	$4G$	2	$r\sqrt{2}$	$r\sqrt{2}$	—	—	
28	$3G$	$G$	$G$	—	2	$r\sqrt{2}$	—	—	0,1	
29	$6G$	$3G$	$G$	$G$	2	—	$r\sqrt{2}$	—	—	$i_{4x} = i_{3x}$
30	$8G$	$G$	$G$	$2G$	—	—	—	—	0,1	

## 8.5. Research of Free Vibrations of Mechanical Systems with One Degree of Freedom

Define frequency and period of small free vibrations of mechanical system with one degree of freedom neglecting forces of resistance and masses of threads.

Derive the equation of motion of a load  $l$ ,  $y = y(t)$ , having accepted for origin position of rest of a load  $l$  (at a static elongation of springs). Determine also amplitude of vibrations of a load  $l$ .

Schemes of systems are shown in Fig. 120—122, and the necessary data are represented in tab. 4.

In the problem following designations are accepted:  $l$  is a load of mass  $m_1$ ; 2 is a block of mass  $m_2$  and radius  $r_2$  (a solid homogeneous disk); 3 is a block of mass  $m_3$  and gyration radius  $i_x$ ; 4 is a solid homogeneous disk of mass  $m_4$  and radius  $r_4$ ; 5 is a disk of mass  $m_5$  and gyration radius  $i'_x$ ; 6 is a thin homogeneous rod of mass  $m_6$  and length  $l$ ; 7 is a rod, which mass is neglected;  $c$  is a spring stiffness factor;  $y_0$  is an initial deflection of a load  $l$  on a vertical from position of the rest corresponding to a static elongation of a spring;  $\dot{y}_0$  is a projection of initial speed  $v_0$  of a load  $l$  on a vertical axis.

In Fig. 120—122 systems of bodies  $l$ —7 are represented in rest position (at a static elongation of springs).

In variants 5, 6, 14 and 23 rod 6 is rigidly connected to a disk 4.

*Example.* It is given:  $m_1 = 1\text{kg}$ ;  $m_2 = 2\text{kg}$ ;  $m_4 = 1\text{kg}$ ;  $m_6 = 3\text{kg}$ ;  $l = 0,6\text{ m}$ ;  $c = 20\text{ N/cm}$ ;  $y_0 = 0,2\text{ cm}$ ;  $\dot{y}_0 = 8\text{ cm/sec}$  (Fig. 119).

Define cyclic frequency  $k$ , the period  $T$  of small free vibrations of system, amplitude  $a$  and derive equation of motion of load  $l$

*Solution.* We will take advantage of the Lagrange's equations for conservative systems. Having accepted for system generalized coordinate a vertical deflection  $y$  of the weight  $l$  from the rest position, which corresponds to the static deflection of a spring, we have

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{y}} \right) - \left( \frac{\partial T}{\partial y} \right) = - \frac{\partial \Pi}{\partial y},$$

where  $T$  is a system kinetic energy;  $\Pi$  is a system potential energy.

Let's calculate kinetic energy  $T$  with the second order infinitesimal accuracy relatively  $\dot{y}$ , and potential energy  $\Pi$  define with the second order infinitesimal accuracy relatively generalized coordinate  $y$ . Determine a kinetic energy of the system as a sum of kinetic energy of bodies  $l$ , 2, 6 and 4:

$$T = T_1 + T_2 + T_6 + T_4.$$

Let's express velocity of the centre of mass of a body 4 and angular velocities of bodies 2, 4 and 6 as a function of the generalized velocity  $\dot{y}$ :

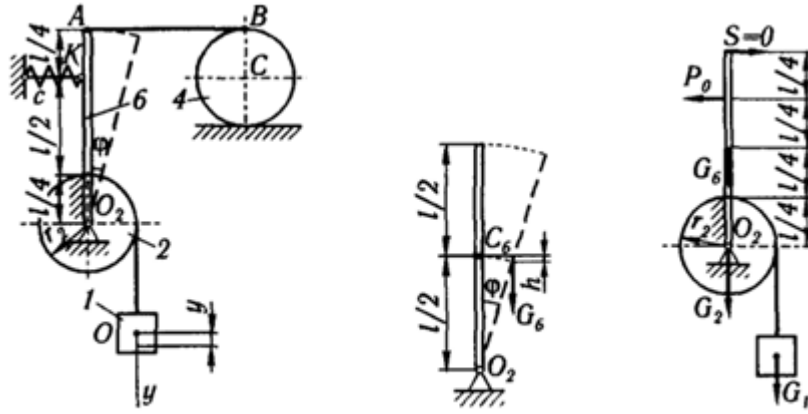


Fig. 119

$$v_1 = \dot{y}; \quad \omega_2 = \dot{y}/r_2; \quad \omega_6 = \omega_2 = \dot{y}/r_2.$$

As we consider small oscillations, then  $v_B = v_A$ . Disk 4 rolls without sliding, so  $v_C = v_B/2$ . Hence,

$$v_C = v_A/2 = \omega_6 l/2 = \omega_2 l/2 = \dot{y}l/(2r_2) = 2\dot{y};$$

$$\omega_4 = v_C/r_4 = 2\dot{y}/r_4.$$

Moment of inertia of the body 4 with respect to the central axis is

$$J_C = m_4 r_4^2 / 2.$$

Moments of inertia of bodies 2 and 6 with respect to rotation axis are

$$J_2 = m_2 r_2^2 / 2; \quad J_6 = m_6 l^2 / 3.$$

Kinetic energy of bodies 1, 2, 4 and 6 is:

$$T_1 = \frac{m_1 v_1^2}{2} = \frac{m_1 \dot{y}^2}{2}; \quad T_2 = \frac{J_2 \omega_2^2}{2} = \frac{m_2 \dot{y}^2}{4};$$

$$T_4 = \frac{m_4 v_C^2}{2} + \frac{J_C \omega_4^2}{2} = 3m_4 \dot{y}^2; \quad T_6 = \frac{J_6 \omega_6^2}{2} = \frac{8m_6 \dot{y}^2}{3}.$$

Thus, kinetic energy of a system is

$$T = T_1 + T_2 + T_6 + T_4 = m_1 \dot{y}^2 / 2 + m_2 \dot{y}^2 / 4 + 8/3 m_6 \dot{y}^2 + 3m_4 \dot{y}^2$$

$$= 1/2 [m_1 + m_2/2 + 16/3 m_6 + 6m_4] \dot{y}^2.$$

Let's calculate a potential energy of a system which equals work of its gravity force and work of elastic force of a spring on a displacement from a deflected location, when load has coordinate  $y$ , to zero position which is a position of rest of a system:

$$\Pi = \Pi_I + \Pi_{II}.$$

Potential energy corresponding to gravity forces on mentioned displacement is

$$\Pi_I = -G_1 y - G_6 h,$$

where  $h$  is a vertical displacement of a center of mass of rod 6, which is computed with the second order infinitesimal accuracy relatively generalized coordinate  $y$ .

It follows from Fig. 119

$$h = l/2 - (l/2) \cos \varphi = (l/2)(1 - \cos \varphi).$$

Decomposition formula for  $\cos \varphi$  is

$$\cos \varphi = 1 - \varphi^2/2! + \varphi^4/4! - \dots$$

Restricting this formula by the two first members and considering that

$$\varphi = y/r_2 = 4y/l,$$

we have

$$\Pi_I = -G_1 y - G_6 \cdot 4y^2/l.$$

The potential energy of the deformed spring is

$$\Pi_{II} = c(f_{st} + \lambda_K)^2/2 - cf_{st}^2/2,$$

where  $f_{st}$  is a static deflection of a spring;  $\lambda_K$  is a displacement of a point of an attachment of a spring  $K$  corresponding to coordinate  $y$ .

As, (Fig. 119),

$$\frac{\lambda_K}{y} = \frac{3/4l}{1/4l'}$$

i.e.,  $\lambda_K = 3y$ , then

$$\Pi_{II} = \frac{c(f_{st} + \lambda_K)^2}{2} - \frac{cf_{st}^2}{2} = 3f_{st}y + \frac{9}{2}cy^2.$$

System potential energy is

$$\Pi = \Pi_I + \Pi_{II} = -G_1 y - (4G_6/l)y^2 + 3cf_{st}y + 9/2cy^2.$$

Since in the position of rest corresponding to a static elongation of a spring

$$(\partial\Pi/\partial y)_{y=0} = 0,$$

then

$$-G_1 + 3cf_{st} = 0.$$

This equation can be obtained also having worked out the equation of moments of forces for the state of static equilibrium of system (Fig. 119):

$$\sum M_{i02} = P_0 \cdot 3/4l - G_1 r_2 = 0,$$

or

$$cf_{st} \cdot 3/4l - G_1 l/4 = 0, \quad \text{i.e., } 3cf_{st} - G_1 = 0.$$

Thus, potential energy of considered mechanical systems is

$$\Pi = 9/2cy^2 - \frac{4G_6}{l}y^2 = 1/2 \left( 9C - \frac{8G_6}{l} \right) y^2.$$

Further,

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{y}} \right) = \left( m_1 + \frac{m_2}{2} + \frac{16}{3}m_6 + 6m_4 \right) \ddot{y};$$

$$\frac{\partial T}{\partial y} = 0; \quad \frac{\partial \Pi}{\partial y} = \left( 9C - \frac{8G_6}{l} \right) y.$$

The Lagrange's equation takes a form

$$\left( m_1 + \frac{m_2}{2} + \frac{16}{3}m_6 + 6m_4 \right) \ddot{y} + \left( 9C - \frac{8G_6}{l} \right) y = 0,$$

or

$$\ddot{y} + \frac{9C - 8G_6/l}{m_1 + m_2/2 + 16m_6/3 + 6m_4} y = 0.$$

Let's introduce designation

$$k^2 = \frac{9C - 8G_6/l}{m_1 + m_2/2 + 16m_6/3 + 6m_4}.$$

Then we shall have the following equation:

$$\ddot{y} + k^2 y = 0.$$

Hence, cyclic frequency of free vibrations is

$$k = \sqrt{\frac{9C - 8G_6/l}{m_1 + m_2/2 + 16m_6/3 + 6m_4}}, \quad k = 27,1 \text{ sec}^{-1}.$$

The period of vibrations is

$$T = 2\pi/k = 2 \cdot 3,14/27,1 = 0,23 \text{ sec}.$$

Integrating the differential equation, we obtain the law of motion of a load  $l$

$$y = C_1 \cos kt + C_2 \sin kt.$$

In order to define constants  $C_1$  and  $C_2$ , determine the equation of velocity of a load

$$\dot{y} = -kC_1 \sin kt + kC_2 \cos kt.$$

Use initial conditions of problem. From the equations  $y = y(t)$  and  $\dot{y} = \dot{y}(t)$  at  $t = 0$  we have

$$y_0 = C_1; \dot{y} = kC_2,$$

whence,

$$C_1 = y_0; C_2 = \dot{y}_0/k.$$

Hence,

$$y = y_0 \cos kt + (\dot{y}_0/k) \sin kt, \\ y = 0,2 \cos 27,1t + 0,3 \sin 27,1t.$$

It is possible to obtain this equation in other form if introduce constants of integration  $a$  and  $\beta$  having designated

$$C_1 = a \sin \beta; C_2 = a \cos \beta.$$

Then

$$y = a \sin(kt + \beta),$$

where  $a = \sqrt{C_1^2 + C_2^2}$ ,  $\beta = \tan(C_1/C_2)$ , or

$$a = \sqrt{y_0^2 + (\dot{y}_0/k)^2}; \beta = \tan(ky_0/\dot{y}_0).$$

Let's calculate numerical values  $a$  and  $\beta$ :  $a = 3,6 \cdot 10^{-2} \text{ m}$ ,  $\beta = \tan 0,68$ .  
As  $\sin \beta > 0$  ( $C_1 > 0$ ), then  $\beta = 34^\circ 12' = 0,597 \text{ radian}$ .

Finally  $y = 3,6 \cdot 10^{-2} \sin(27,1t + 0,597) \text{ m}$ .

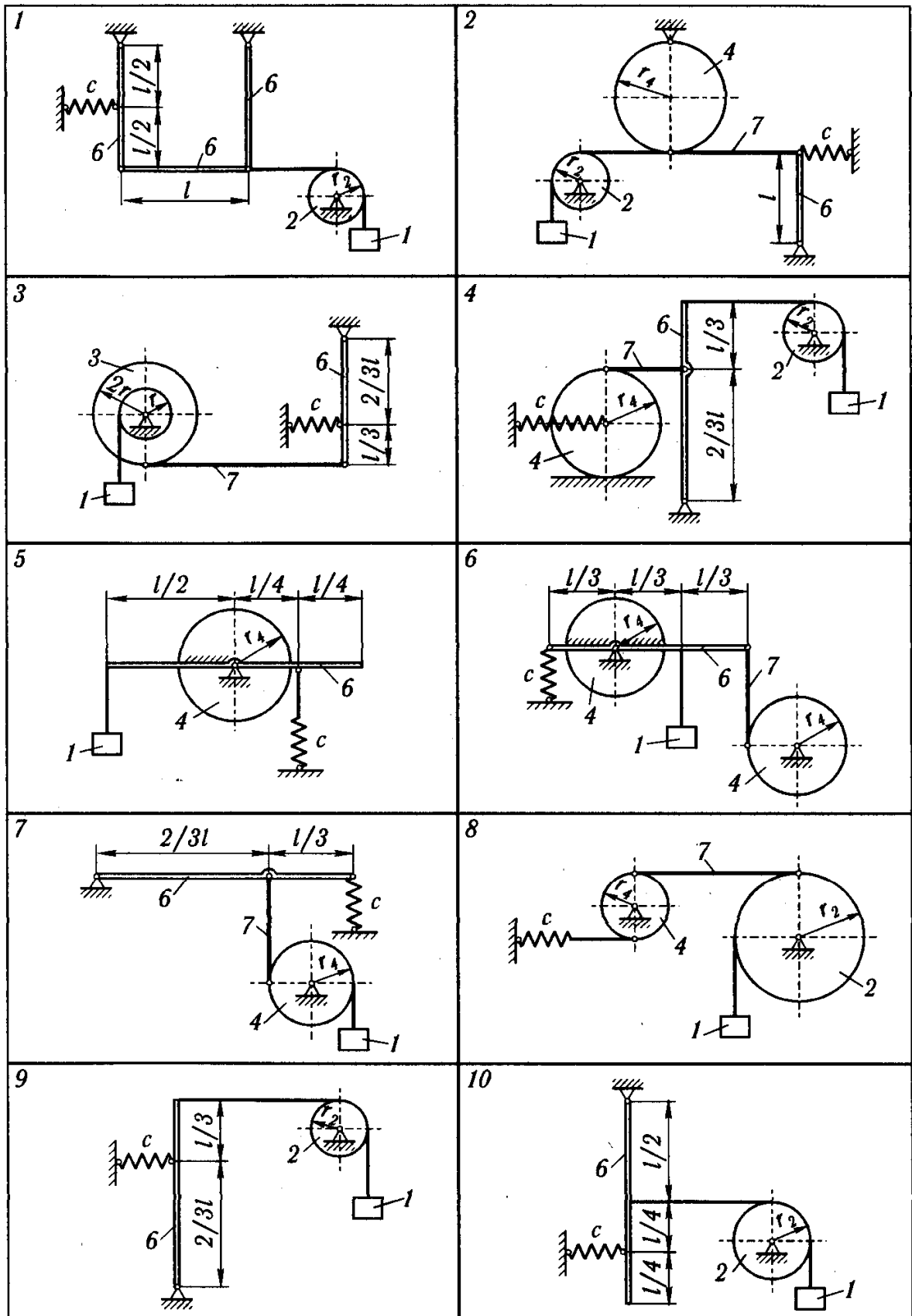


Fig. 120

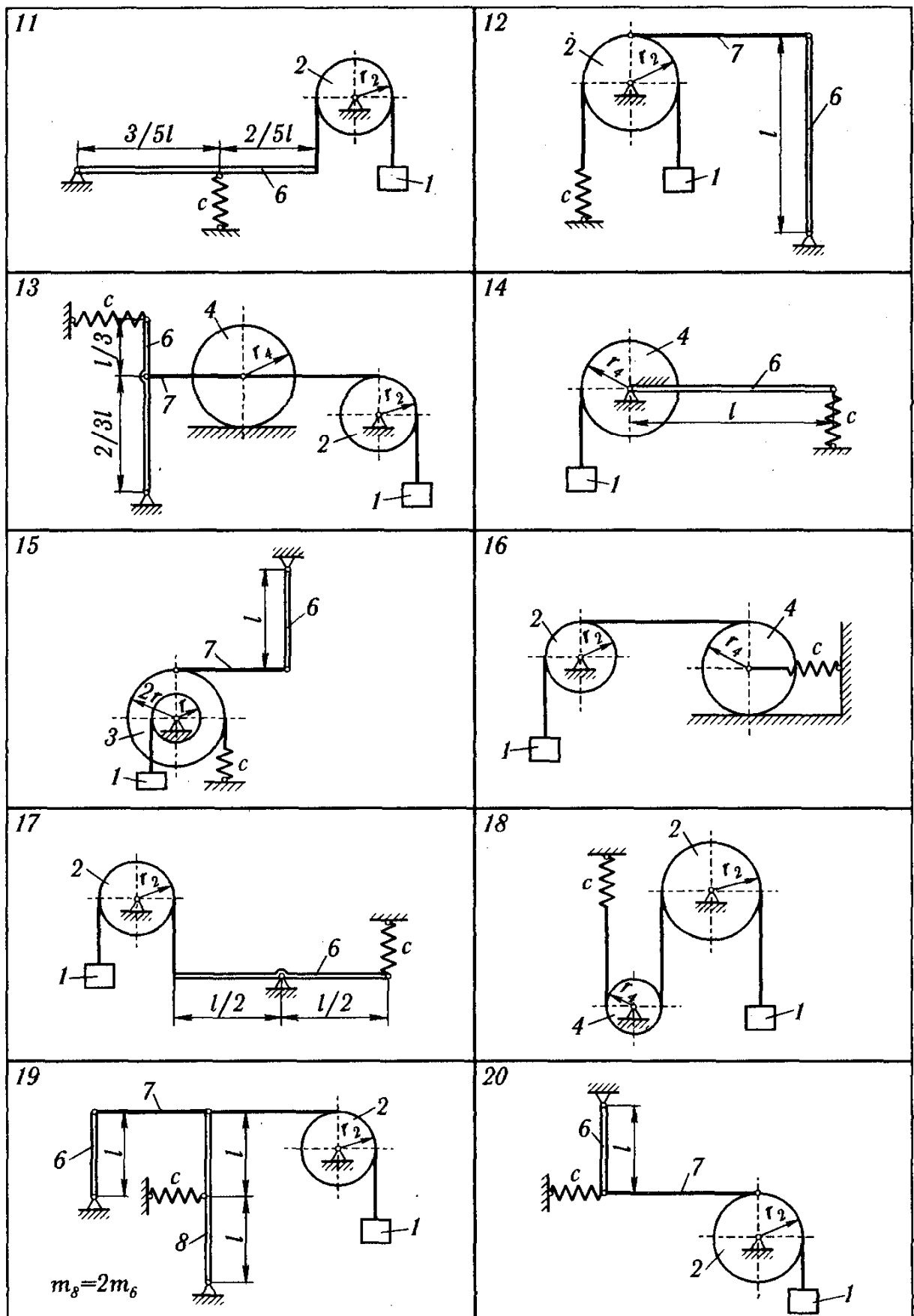


Fig. 121



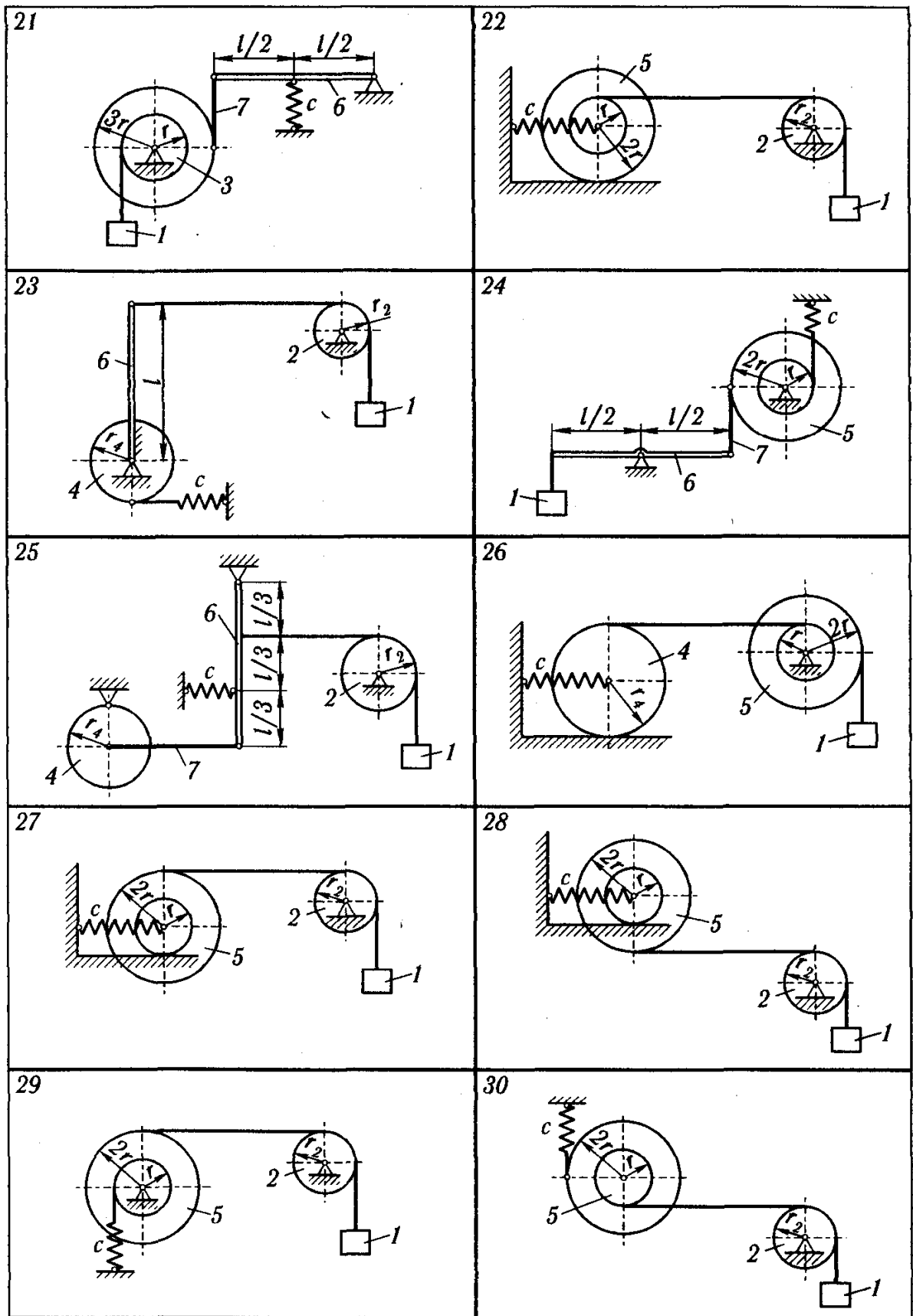


Fig. 122

Tab. 4

Variant number (Fig. 120-122)	$l$	$i_x$	$i'_x$	$r_4$	$m_1$	$m_2$	$m_3,$ $m_4,$ $m_5$	$m_6$	$c$	Initial conditions ( $t = 0$ )	
	$m$				$kg$				$N/cm$	$y_0,$ $cm$	$\dot{y}_0,$ $m/sec$
1	0,5	-	-	-	1	2	-	3	40	0,1	5,0
2	0,5	-	-	0,2	1	2	2	3	40	0	6,0
3	0,5	$3/2r$	-	-	1	-	4	3	20	0,2	7,0
4	0,6	-	-	-	1	2	3	2	36	0,2	0
5	0,6	-	-	0,15	1	-	3	3	16	0	8,0
6	0,6	-	-	0,15	1	-	1	1	40	0,3	7,0
7	-	-	-	-	1	-	2	2	40	0,4	0
8	-	-	-	-	1	3	2	-	40	0	6,0
9	0,6	-	-	-	1	2	-	3	38	0,5	5,0
10	0,6	-	-	-	1	2	-	3	32	0	6,0
11	-	-	-	-	1	2	-	3	30	0,4	7,0
12	0,5	-	-	-	1	2	-	3	20	0,2	0
13	0,3	-	-	-	1	1	1	2	32	0	8,0
14	0,4	-	-	0,1	1	-	2	3	20	0	7,0
15	0,4	$r\sqrt{3}$	-	-	1	-	2	2	20	0,1	0
16	-	-	-	-	1	2	3	-	32	0,3	6,0
17	-	-	-	-	1	2	-	2	20	0	5,0
18	-	-	-	-	1	2	1	-	40	0	6,0
19	0,2	-	-	-	1	1	-	1	32	0,1	0
20	0,5	-	-	-	1	2	-	3	20	0,4	7,0
21	-	$2r$	-	-	1	-	2	3	32	0	8,0
22	-	-	$r\sqrt{2}$	-	1	2	4	-	40	0,1	7,0
23	0,4	-	-	0,2	1	2	2	3	40	0,3	0
24	-	-	$r\sqrt{3}$	-	1	-	3	2	40	0	6,0
25	0,3	-	-	0,1	1	2	2	1	40	0,2	5,0
26	-	$r\sqrt{2}$	-	-	1	-	2	-	40	0,3	0
27	-	-	$3r/2$	-	1	2	3	-	40	0	6,0
28	-	-	$r\sqrt{3}$	-	1	2	3	-	40	0,2	0
29	-	-	$4r/3$	-	1	2	3	-	40	0	7,0
30	-	-	$r\sqrt{2}$	-	1	2	3	-	40	0,3	7,0

## 8.6. Application of the Lagrange's Equations to Research of Motion of Mechanical System with Two Degrees of Freedom

The mechanical system of bodies (Fig. 127—129) moves under the action of constant forces  $\mathbf{P}$  and couples with the moments  $M$  or only under the gravity forces.

Make up the equations of motion of system in generalized coordinates  $q_1$  and  $q_2$  at the specified initial conditions. The necessary data are reduced in tab. 5; in the same place recommended generalized coordinates are specified ( $x$  and  $\varphi$  are generalized coordinates for absolute motion, and  $\xi$  is for relative motion).

Neglect masses of the threads. Take into consideration that the rolling of wheels occurs without a slippage. Neglect rolling friction and forces of resistance in bearings. Wheels for which in the table 5 inertia radii are not specified, consider as solid homogeneous disks. Consider cages (cranks) as thin homogeneous rods. Accept that in variants 6, 9, 11, 20, 22 and 30 mechanism is located in a horizontal plane.

Radii of gyration of bodies 2 and 3 are defined with respect to the central axis perpendicular to the figure. Coefficient of viscosity is a quantity  $b$  in expression  $\mathbf{R} = -b\mathbf{v}$ , where  $\mathbf{v}$  is a relative velocity of bodies 1 and 2

*Example* It is given: masses of bodies of mechanical system (Fig. 123)  $m_1 = 3m$ ;  $m_2 = 8m$ ;  $m_3 = m_4 = m_6 = 2m$ ;  $m_5 = 4m$ ;  $m_7 = m$ ;  $\mathbf{P}$  is a constant force applied to a body 2;  $M$  is a constant moment applied to a block 6;  $b$  is a coefficient of proportionality in expression for force of resistance to motion of a body 5:  $\mathbf{R} = -b\mathbf{v}_5$  ( $\mathbf{v}_5$  is a velocity of a body 5);  $L$  is a length of a thread 3;  $r$  is a radius of the blocks 4 and 6. Here the thread 3 is accepted ponderable. This is a

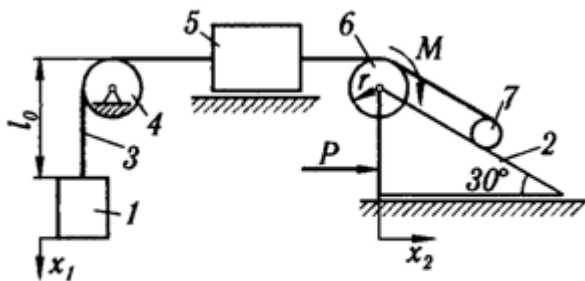


Fig. 123

complicating in comparison with the common condition of the problem. Thread sagging is not considered.

Consider all wheels as solid homogeneous disks. Neglect sliding friction of a body 2.

Make up the equations of motion of system in generalized coordinates  $q_1 = x_1$ ;  $q_2 = x_2$ .

Initial conditions:  $q_{10} = 0$  (initial distance on a vertical from the lower end of a thread 3 to its horizontal site equals  $l_0$ ),  $q_{20} = 0$ ,  $\dot{q}_{10} = 0$ ,  $\dot{q}_{20} = \dot{x}_{20}$ .

In Fig. 123 system is figured in initial position.

*Solution.* In order to resolve problem, we will apply the Lagrange's equations

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{x}_1} - \frac{\partial T}{\partial x_1} = Q_1;$$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{x}_2} - \frac{\partial T}{\partial x_2} = Q_2,$$

where  $T$  is a kinetic energy of a system;  $Q_1$  and  $Q_2$  are generalized forces corresponding to generalized coordinates  $x_1$  and  $x_2$ .

For the given system  $T = \sum_{i=1}^7 T_i$ .

Let's express velocities of the centers of mass of rigid bodies of system through

generalized velocities:

$$\left. \begin{aligned} v_1 &= v_5 = \dot{x}_1; \\ v_2 &= v_6 = \dot{x}_2; \\ v_7 &= \frac{(\dot{x}_1 + \dot{x}_2)^2}{4} + \dot{x}_2^2 - (\dot{x}_1 + \dot{x}_2)\dot{x}_2 \cos \alpha, \end{aligned} \right\}$$

Taking into account that  $\alpha = 30^\circ$ ,

$$v_7^2 = 0,25[\dot{x}_1^2 + (5 - 2\sqrt{3})2\dot{x}_2^2 - 2(\sqrt{3} - 1)\dot{x}_1\dot{x}_2].$$

It was considered that  $(\dot{x}_1 + \dot{x}_2)/2$  is a velocity of the centre of mass of a body 7 concerning a body 2, i.e., relative velocity and  $\dot{x}_2$  is its transport velocity (Fig. 124).

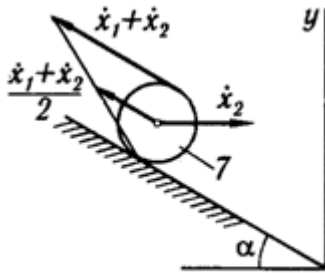


Fig. 124

Angular velocities of bodies (Fig. 123—125)

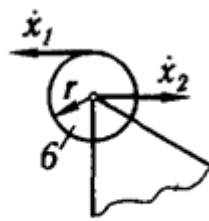


Fig. 125

$$\left. \begin{aligned} \omega_4 &= \frac{\dot{x}_1}{r}; \\ \omega_6 &= \omega_7 = \frac{\dot{x}_1 + \dot{x}_2}{r}. \end{aligned} \right\}$$

Moments of inertia of wheels with respect to the central axes are

$$J_4 = J_6 = \frac{2mr^2}{2} = mr^2; J_7 =$$

$$(m/2)(r/2)^2 = mr^2/8.$$

Kinetic energy of bodies 1, 2, 4—7 is

$$T_1 = m_1 v_1^2 / 2 = 3m \dot{x}_1^2 / 2; T_2 = m_2 v_2^2 / 2 = 4m \dot{x}_2^2;$$

$$T_4 = \frac{J_4 \omega_4^2}{2} = \frac{m \dot{x}_1^2}{2}; T_5 = \frac{m_5 v_3^2}{2} = 2m \dot{x}_1^2;$$

$$T_6 = \frac{m_6 v_6^2}{2} + \frac{J_6 \omega_6^2}{2} = 0,5m(\dot{x}_1^2 + 3\dot{x}_2^2 + 2\dot{x}_1\dot{x}_2);$$

$$T_7 = \frac{m_7 v_7^2}{2} + \frac{J_7 \omega_7^2}{2} = (m/16)[3\dot{x}_1^2 + (11 - 4\sqrt{3})\dot{x}_2^2 - 2(2\sqrt{3} - 3)\dot{x}_1\dot{x}_2].$$

Taking into account that all points of a thread 3 have equal velocities  $v_{3i} = v_3 = \dot{x}_1$ , and that  $\sum m_{3i} = m$ , we have  $T_3 = m_3 v_3^2 / 2 = m \dot{x}_1^2$ .

Substituting all these magnitudes, one can obtain

$$T = \left(\frac{m}{16}\right)[75\dot{x}_1^2 + (99 - 4\sqrt{3})\dot{x}_2^2 + 2(11 - 2\sqrt{3})\dot{x}_1\dot{x}_2].$$

And now let's define generalized forces  $Q_1$  and  $Q_2$  corresponding to generalized coordinates  $x_1$  and  $x_2$ . Consider virtual work of all forces on virtual displacements  $\delta x_1$  and  $\delta x_2$ .

Define virtual work of all the forces done in a virtual displacement  $\delta x_1$ . It should be born in mind that now  $\delta x_2 = 0$ . We have

$$\delta A_1 = \delta A(m_1 \mathbf{g}) + \delta A(\mathbf{R}) + \delta A(M) + \delta A(m_7 \mathbf{g}) + \delta A(m_3 \mathbf{g}).$$

Calculate all of these members:

$$\delta A(m_1 \mathbf{g}) = m_1 g \delta x_1, \quad \delta A(\mathbf{R}) = -b v_5 \delta x_1 = -b \dot{x}_1 \delta x_1, \quad \delta A(M) = -\frac{M}{r} \delta x_1,$$

$$\delta A(m_7 \mathbf{g}) = -\frac{m_7 g}{2} \sin 30^\circ \delta x_1.$$

Let's notice that work of a gravity force  $m_7 \mathbf{g}$  of a thread 3 from position  $a'b'$  to position  $ab$ , at which  $x_1 = 0$ , is equal to the work of a gravity force of a site of a thread  $bb'$  at its displacement to position  $aa'$  (Fig. 126).

Thus,

$$\delta A(m_3 \mathbf{g}) = \frac{m_3 g}{L} (x_1 + \frac{l_0}{2}) \delta x_1.$$

Then, taking into account the data of the problem, we receive

$$\delta A_1 = \left[ 2,75mg - b\dot{x}_1 - \frac{M}{r} + \frac{2mg}{L} (x_1 + \frac{l_0}{2}) \right] \delta x_1.$$

Hence,

$$Q_1 = 2,75mg - b\dot{x}_1 - \frac{M}{r} + \frac{2mg}{L} (x_1 + \frac{l_0}{2}).$$

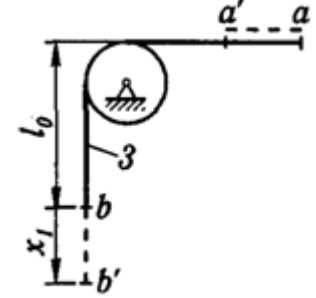


Fig. 126

Define virtual work of all the forces done in a virtual displacement  $\delta x_2$ . It should be born in mind that now  $\delta x_1 = 0$ . So we have

$$\delta A_2 = \delta A(\mathbf{P}) + \delta A(m_7 \mathbf{g}) + \delta A(M).$$

Determine all of these members:

$$\delta A(\mathbf{P}) = P \delta x_2, \quad \delta A(m_7 \mathbf{g}) = -\frac{m_7 g}{2} \sin 30^\circ \delta x_2, \quad \delta A(M) = -\frac{M}{r} \delta x_2.$$

Thus,

$$\delta A_2 = \left( P - \frac{mg}{4} - \frac{M}{r} \right) \delta x_2,$$

whence

$$Q_2 = P - \frac{mg}{4} - \frac{M}{r}.$$

Substituting all results in the Lagrange's equations, we receive the differential equations of motion of a system:

$$\frac{75}{8} m \ddot{x}_1 + \frac{11 - 2\sqrt{3}}{8} m \ddot{x}_2 = 2mg \frac{x_1}{L} + mg \left( 2,75 + \frac{2l_0}{L} \right) - b\dot{x}_1 - \frac{M}{r};$$

$$\frac{99 - 4\sqrt{3}}{8} m \ddot{x}_2 + \frac{11 - 2\sqrt{3}}{8} m \ddot{x}_1 = -\frac{mg}{4} + P - \frac{M}{r}.$$

Expressing  $\ddot{x}_2$  from the second equation and substituting in the first one, we obtain

$$\ddot{x}_1 + 2n\dot{x}_1 - cx_1 = a,$$

where

$$n = \frac{0,0538b}{m}; \quad c = \frac{0,215g}{L};$$

$$a = g(0,298 + 0,215l_0/L) - 0,099M/(rm) - 0,0088P/m.$$

Let's define the solution of this linear non-uniform differential equation. Its characteristic equation is

$$z^2 + 2nz - c = 0.$$

Its equation roots are

$$z_{1,2} = -n \pm \sqrt{n^2 + c}.$$

Then the general solution of the differential equation has a form:

$$x_1 = e^{-nt} \left( C_1 e^{\sqrt{n^2 + ct}} + C_2 e^{-\sqrt{n^2 + ct}} \right) - \frac{a}{c}.$$

In order to define constants  $C_1$  and  $C_2$ , differentiate this solution

$$\dot{x}_1 = e^{-nt} \left[ \left( -n + \sqrt{n^2 + c} \right) C_1 e^{\sqrt{n^2 + ct}} - \left( n + \sqrt{n^2 + c} \right) C_2 e^{-\sqrt{n^2 + ct}} \right].$$

Using initial conditions: at  $t = 0$ ,  $x_1 = 0$ ;  $\dot{x}_1 = 0$ , we have

$$\begin{aligned} C_1 + C_2 - a/c &= 0; \\ \left( -n + \sqrt{n^2 + c} \right) C_1 - \left( n + \sqrt{n^2 + c} \right) C_2 &= 0, \end{aligned}$$

whence

$$\left. \begin{aligned} C_1 &= \frac{a}{2c + \sqrt{n^2 + c}} \left( n + \sqrt{n^2 + c} \right); \\ C_2 &= \frac{a}{2c + \sqrt{n^2 + c}} \left( -n + \sqrt{n^2 + c} \right). \end{aligned} \right\}$$

Thus, we have equation of motion of the system describing change of the first generalized coordinate. To receive the second equation of motion, we find

$$\ddot{x}_2 = \frac{8}{99-4\sqrt{3}} \left( -\frac{g}{4} + \frac{P}{m} - \frac{M}{m \cdot r} \right) - \frac{11-2\sqrt{3}}{99-4\sqrt{3}} \ddot{x}_1, \text{ or } \ddot{x}_2 = k - 0,0818\ddot{x}_1,$$

where  $k = 0,0869 \left[ \frac{1}{m} \left( P - \frac{M}{r} \right) - \frac{g}{4} \right]$ .

Integrating, we obtain

$$\dot{x}_2 = kt - 0,0818\dot{x}_1 + C_3; \quad x_2 = kt^2/2 - 0,0818x_1 + C_3t + C_4.$$

Using initial conditions: at  $t = 0$ ,  $x_1 = 0$ ;  $x_2 = 0$ ;  $\dot{x}_1 = 0$ ;  $\dot{x}_2 = \dot{x}_{20}$ , we find  $C_3 = \dot{x}_{20}$ ;  $C_4 = 0$ .

Finally we have

$$x_2 = \frac{kt^2}{2} - 0,0818 \left[ e^{-nt} \left( C_1 e^{\sqrt{n^2 + ct}} + C_2 e^{-\sqrt{n^2 + ct}} \right) - \frac{a}{c} \right] + \dot{x}_{20}t.$$

This is the second equation of motion of a system.

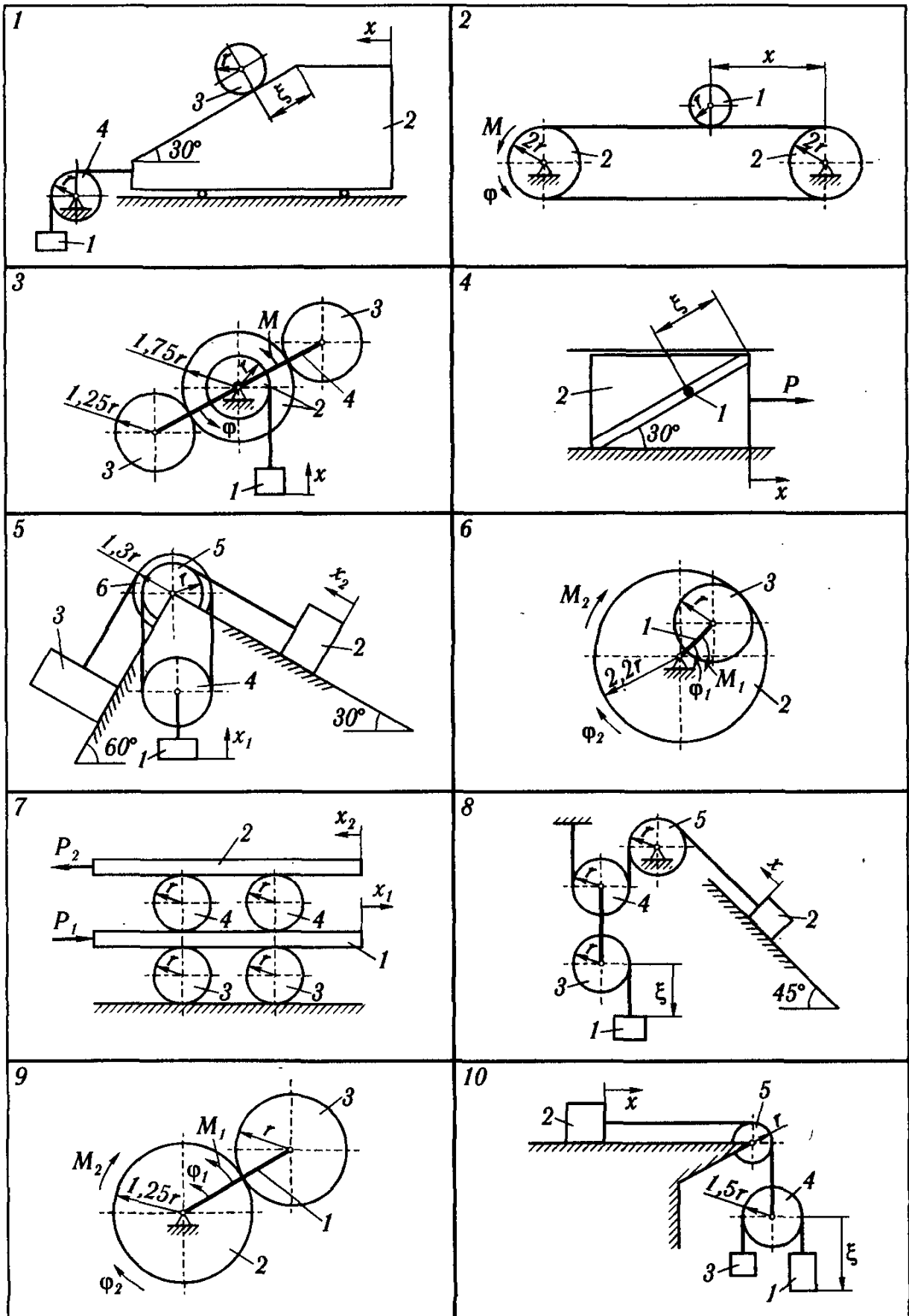


Fig. 127

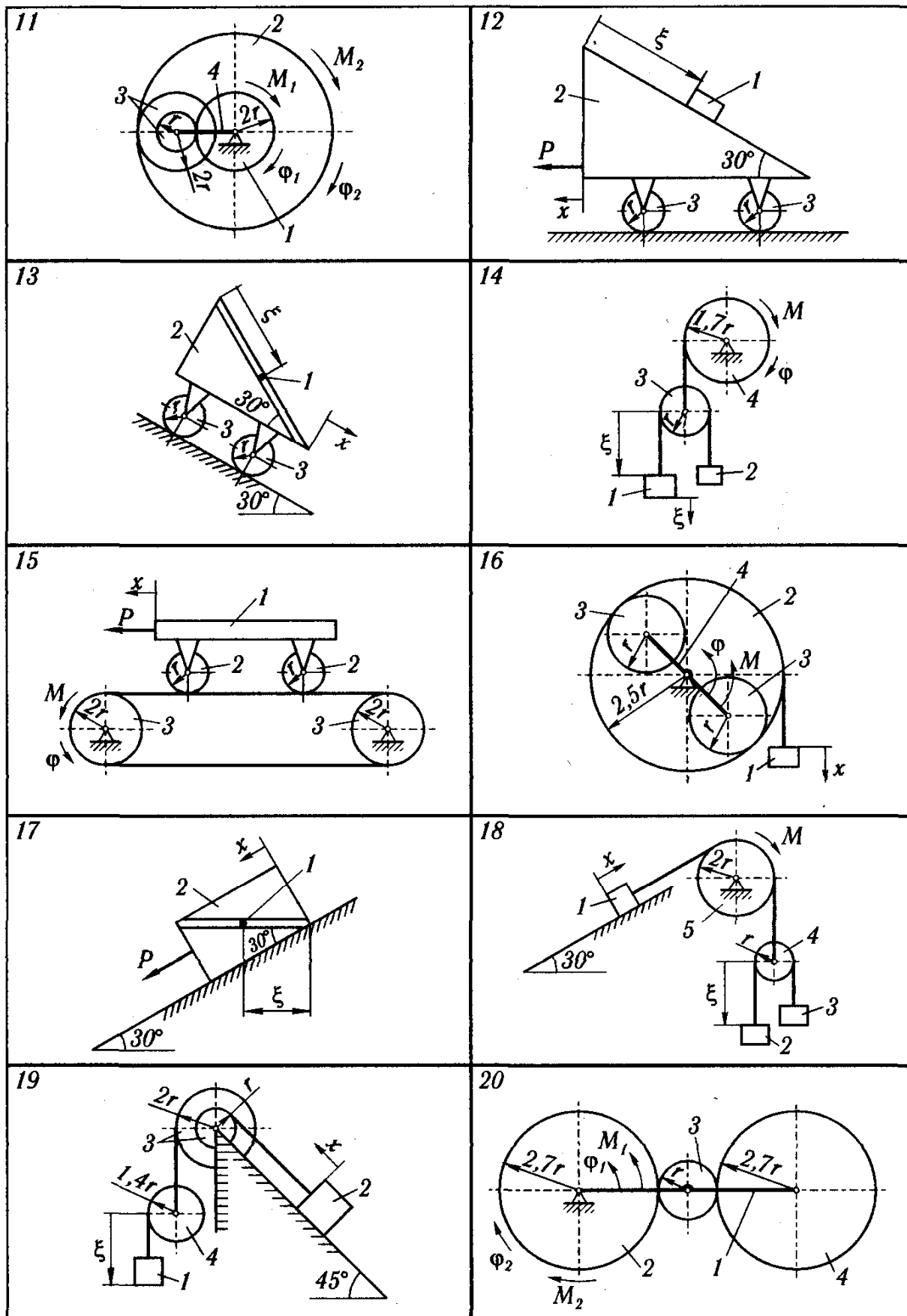


Fig. 128



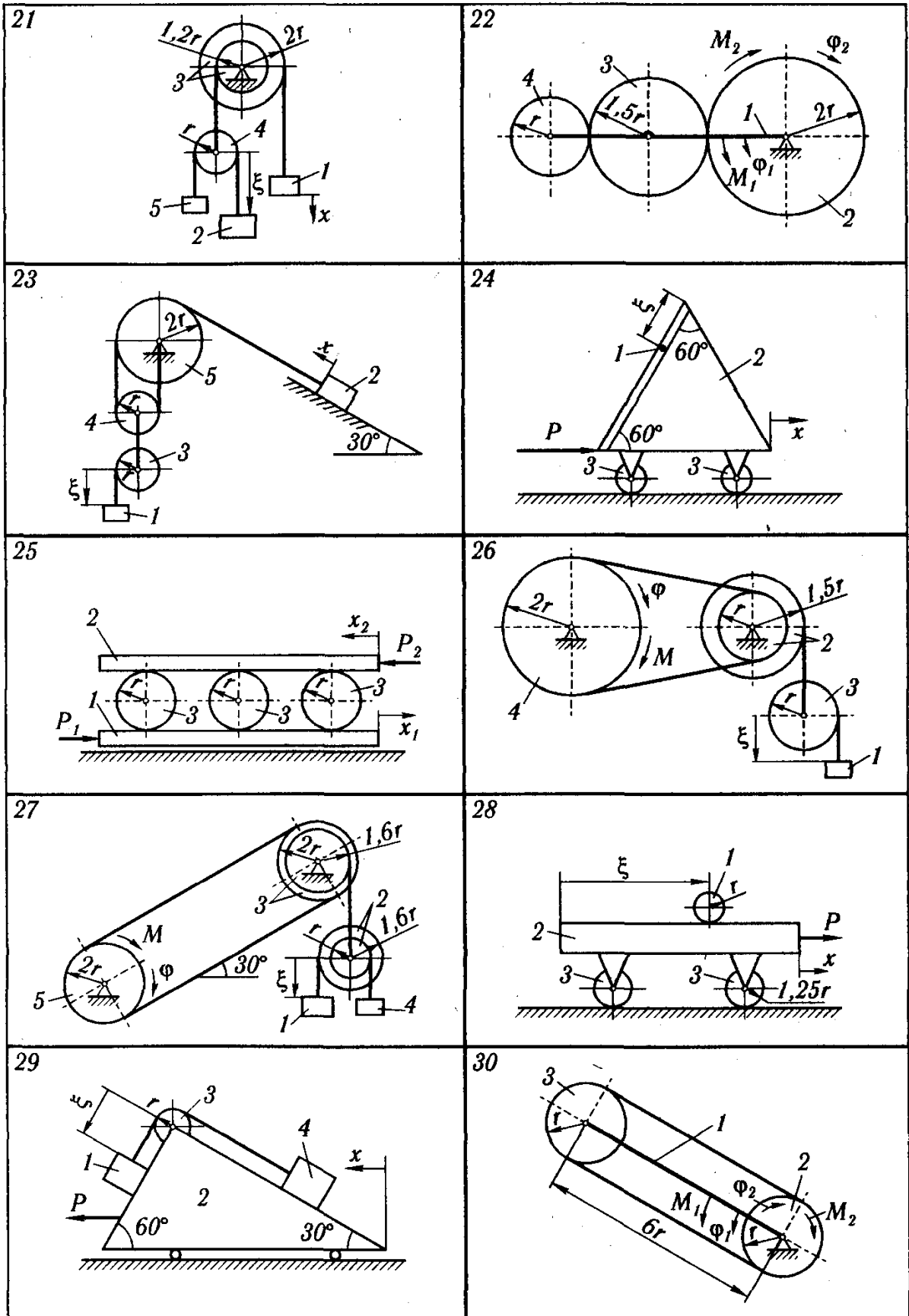


Fig. 129

Tab. 5

Variant number (Fig. 127-129)	Masses of bodies					Radii of gyration		Forces $P$	Moments $M$	Coefficients of		Generalized coordinates		Initial conditions				Supplementary data
	1	2	3	4	5	$i_{2y}$	$i_{3y}$			friction	viscosity	$q_1$	$q_2$	$q_{10}$	$q_{20}$	$\dot{q}_{10}$	$\dot{q}_{20}$	
1	$2m$	$6m$	$m$	$m$	—	—	—	—	—	—	$x$	$\xi$	0	0	0	0		
2	$m$	$3m$	—	—	—	—	—	—	$M$	—	$\varphi$	$x$	0	0	0	0	Neglect mass of the belt	
3	$m$	$3m$	$2m$	—	—	$r\sqrt{2}$	—	—	$M$	—	$\varphi$	$x$	0	0	0	0	Moment $M$ is applied at cage	
4	$m$	$4m$	—	—	—	—	—	—	—	0	$x$	$\xi$	0	0	$\dot{x}_0$	0	Body $I$ is a particle	
5	$m$	$2m$	$4m$	$2m$	$2m$	—	—	—	—	$f$	$x_1$	$x_2$	0	0	0	0	Pulleys 5 and 6 are freely shafted on a common axle, their masses being equal	
6	$m$	$2m$	$3m$	—	—	$2r$	—	—	$M_1; M_2$	—	$\varphi_1$	$\varphi_2$	0	0	0	0	Moment $M_1$ is applied at cage	
7	$3m$	$3m$	$m$	$m$	—	—	—	$P_1; P_2$	—	—	$x_1$	$x_2$	0	0	0	0		
8	$m$	$2m$	$2m$	$2m$	—	—	—	—	—	$f$	$x$	$\xi$	0	0	0	$\dot{\xi}_0$		
9	$m$	$2m$	$3m$	—	—	—	—	—	$M_1; M_2$	—	$\varphi_1$	$\varphi_2$	0	0	0	0	Moment $M_1$ is applied at cage	
10	$2m$	$2m$	$m$	$2m$	$m$	—	—	—	—	$f$	$x$	$\xi$	0	0	$\dot{x}_0$	0		
11	$m$	$3m$	$2m$	$m$	—	$r\sqrt{2}$	—	—	$M_1; M_2$	—	$\varphi_1$	$\varphi_2$	0	0	0	0		
12	$2m$	$5m$	$m$	—	—	—	—	$P$	—	$f$	$x$	$\xi$	0	0	0	0		
13	$m$	$3m$	$2m$	—	—	—	—	—	—	—	$x$	$\xi$	0	0	0	0	Body $I$ is a particle	
14	$2m$	$m$	$m$	$2m$	—	—	—	—	$M$	—	$\varphi$	$\xi$	0	0	0	$\dot{\xi}_0$		
15	$3m$	$m$	$2m$	—	—	—	—	$P$	$M$	—	$\varphi$	$x$	0	0	0	0	Neglect mass of the belt	

Variant number (Fig. 127-129)	Masses of bodies					Radii of gyration		Forces $P$	Moments $M$	Coefficients of		Generalized coordinates		Initial conditions				Supplementary data
	1	2	3	4	5	$i_{2y}$	$i_{3y}$			friction	viscosity	$q_1$	$q_2$	$q_{10}$	$q_{20}$	$\dot{q}_{10}$	$\dot{q}_{20}$	
16	$2m$	$3m$	$2m$	$m$	—	$2r$	—	—	$M$	—	$\varphi$	$x$	0	0	0	$\dot{x}_0$	Moment $M$ is applied at cage	
17	$m$	$3m$	—	—	—	—	—	$P$	—	$x$	$\xi$	0	$\xi_0$	0	0	0	Body $l$ is a particle	
18	$2m$	$2m$	$m$	$m$	$3m$	—	—	—	$M$	—	$x$	$\xi$	0	0	0	0		
19	$2m$	$2m$	$3m$	$m$	—	—	$r\sqrt{2}$	—	—	$x$	$\xi$	0	0	$\dot{x}_0$	0	0		
20	$2m$	$3m$	$m$	$3m$	—	—	—	—	$M_1; M_2$	—	$\varphi_1$	$\varphi_2$	0	0	0	0	Moment $M_1$ is applied at cage	
21	$2m$	$2m$	$3m$	$2m$	$m$	—	$r\sqrt{2}$	—	—	$x$	$\xi$	0	0	$\dot{x}_0$	0	0		
22	$m$	$3m$	$2m$	$m$	—	—	—	—	$M_1; M_2$	—	$\varphi_1$	$\varphi_2$	0	0	0	0	The same	
23	$2m$	$m$	$m$	$m$	$3m$	—	—	—	—	$x$	$\xi$	0	0	0	0	$\dot{\xi}_0$		
24	$m$	$3m$	$m$	—	—	—	—	$P$	—	$x$	$\xi$	0	$\xi_0$	0	0	0	Body $l$ is a particle	
25	$2m$	$2m$	$m$	—	—	—	—	$P_1; P_2$	—	$x_1$	$x_2$	0	0	0	0	0		
26	$m$	$3m$	$2m$	$3m$	—	—	$r$	—	$M$	$\varphi$	$\xi$	0	0	0	0	0		
27	$2m$	$2m$	$3m$	$m$	$2m$	$r\sqrt{2}$	$r\sqrt{3}$	—	$M$	$\varphi$	$\xi$	0	0	0	0	0		
28	$m$	$3m$	$m$	—	—	—	—	$P$	—	$x$	$\xi$	0	0	0	0	0		
29	$2m$	$4m$	$m$	$m$	—	—	—	$P$	—	$x$	$\xi$	0	0	$\dot{x}_0$	0	0		
30	$3m$	$2m$	$2m$	—	—	—	—	—	$M_1; M_2$	$\varphi_1$	$\varphi_2$	0	0	$\dot{\varphi}_{10}$	0	0	Moment $M_1$ is applied at cage	

## REFERENCES

1. S. Targ. Theoretical Mechanics. A Short Course. – Moscow: Foreign Languages Publishing House, – 421 p.
2. Яблонский А.А. и др. Сборник заданий для курсовых работ по теоретической механике. – М.: Высш. шк., 1978. – 240 с.
3. Тарг. С.М. Краткий курс теоретической механики. – М.: Наука, 1974. – 400 с.
4. J. M. Krodkiewski. Dynamics of Mechanical Systems. – University of Melbourne, Department of Mechanical and Manufacturing Engineering, 2008. – 199 p.
5. J. H. Williams. Fundamentals of Applied Dynamics. – New York, Wiley, 1996
6. I. Percival, D. Richards, Introduction to Dynamics. – Cambridge University Press, 1982
7. J.L. Synge, B.A. Griffiths. Principles of Mechanics. – McGraw-Hill, New York, 1959
8. S. Neil Rasband. Dynamics. – New York, Wiley, 1983
9. E.T. Whittaker. A Treatise on the Analytical Dynamics of Particles and Rigid Bodies. – Cambridge University Press, Cambridge UK, 1927

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**Долгов Олександр Михайлович**

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